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IDEALS OF SQUARE SUMMABLE POWER SERIES IN SEVERAL VARIABLES

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ABSTRACT. Let $\mathscr{C}(z)$ be the Hilbert space of formal power series in $z_1, \cdots, z_r$ ($r \geqq 1$). An ideal of $\mathscr{C}(z)$ is a vector subspace $\mathscr{M}$ of $\mathscr{C}(z)$ which contains $z_1 f(z), \cdots, z_r f(z)$ whenever it contains $f(z)$. If $B(z)$ is a formal power series such that $B(z)f(z)$ belongs to $\mathscr{C}(z)$ and $\|B(z)f(z)\| = \|f(z)\|$, then the set $\mathscr{M}(B)$ of all products $B(z)f(z)$ is a closed ideal of $\mathscr{C}(z)$. In the case $r=1$, Beurling showed that every closed ideal is of this form for some such $B(z)$. Here we give conditions under which a closed ideal is of the form $\mathscr{M}(B)$ for $r \geqq 2$.

By a formal power series in $r \geqq 1$ variables we mean a formal sum $f(z) = \sum a_n z^n$, where $a_n = a_{n_1 \cdots n_r}$ is a complex number depending on the multi-index $n = n_1 \cdots n_r$ and $z^n = z_1^{n_1} \cdots z_r^{n_r}$ ($n_1, \cdots, n_r = 0, 1, 2, \cdots$). Questions of convergence do not arise because we treat $z_1, \cdots, z_r$ as indeterminates. Let $\mathscr{C}(z)$ be the Hilbert space of formal power series $f(z) = \sum a_n z^n$ such that

$$\|f(z)\|^2 = \sum |a_n|^2 < \infty.$$

An ideal of $\mathscr{C}(z)$ is a vector subspace $\mathscr{M}$ of $\mathscr{C}(z)$ which contains $z_1 f(z), \cdots, z_r f(z)$ whenever it contains $f(z)$. Except for the case $r=1$, the closed ideals of $\mathscr{C}(z)$ are not known. For an example of a closed ideal, let $B(z)$ be a formal power series such that the formal product $B(z)f(z)$ belongs to $\mathscr{C}(z)$ and $\|B(z)f(z)\| = \|f(z)\|$ for every $f(z)$ in $\mathscr{C}(z)$. Then the set $\mathscr{M}(B)$ of all products $B(z)f(z)$, where $f(z)$ is in $\mathscr{C}(z)$, is a closed ideal of $\mathscr{C}(z)$. In the case $r=1$, it is a well-known fact that every closed ideal is of this form for some such series $B(z)$. The theorem was given by Beurling [**1**]. We are following the notation of Rovnyak ([**4**]; or see deBranges and Rovnyak [**2**, p. 12]) whose elementary proof of Beurling's theorem is here extended in a straightforward manner. The resulting theorem gives conditions under which a closed ideal is of the form $\mathscr{M}(B)$ for $r \geqq 2$.

LEMMA 1. *Let $B(z)$ be a formal power series. A necessary and sufficient condition that $B(z)f(z)$ belong to $\mathscr{C}(z)$ and $\|B(z)f(z)\| = \|f(z)\|$ for every $f(z)$ in $\mathscr{C}(z)$ is that $(z^n B(z))$ be an orthonormal set in $\mathscr{C}(z)$. In this case the*

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expansion $B(z)f(z)=\sum a_n z^n B(z)$, *where* $f(z)=\sum a_n z^n$, *is valid in the metric of* $\mathscr{C}(z)$.

LEMMA 2. *Let* $f(z)$ *and* $g(z)$ *be formal power series such that* $(z^n f(z))$ *and* $(z^n g(z))$ *are orthonormal sets in* $\mathscr{C}(z)$. *If* $\langle z^m f(z), z^n g(z)\rangle = 0$ *whenever* $m \neq n$, *then* $f(z)$ *and* $g(z)$ *are linearly dependent.*

THEOREM. *If* $\mathscr{M}$ *is a closed ideal of* $\mathscr{C}(z)$ *containing a nonzero element, let* P_k *denote the projection of* $\mathscr{M}$ *into the set of series of the form* $z_k f(z)$ *with* $f(z)$ *in* $\mathscr{M}$ $(k=1, \cdots, r)$. *A necessary and sufficient condition that* $\mathscr{M} = \mathscr{M}(B)$ *for some series* $B(z)$ *such that* $(z^n B(z))$ *is an orthonormal set in* $\mathscr{C}(z)$ *is that for every* $f(z)$ *in* $\mathscr{M}$ *the commutativity relations* $P_i z_j f(z) = z_j P_i f(z)$ *(with* $i, j = 1, \cdots, r$ *and* $i \neq j$*) be valid. When* $B(z)$ *exists, it is unique to within a multiplicative constant of absolute value one.*

PROOF OF LEMMA 1. As in [4], it is convenient to introduce the notion of formal convergence. A sequence $(f_j(z))$ of formal power series in $\mathscr{C}(z)$ is said to converge formally if for each multi-index $j = j_1 \cdots j_r$ $(j_k = 0, 1, 2, \cdots; k = 1, \cdots, r)$ the corresponding sequence of jth coefficients converges. If a sequence converges in the metric of $\mathscr{C}(z)$, then it converges formally to the same limit.

Let $B(z)$ meet the conditions of the lemma. It is then easy to see that the series $\sum a_n z^n B(z)$ converges in the metric of $\mathscr{C}(z)$, its formal sum being $B(z)f(z)$. It follows that the equation $B(z)f(z) = \sum a_n z^n B(z)$ is valid in $\mathscr{C}(z)$, i.e. that $B(z)f(z)$ belongs to $\mathscr{C}(z)$ as claimed. The sufficiency proof is complete upon our noting that

$$\begin{aligned}\|B(z)f(z)\|^2 &= \left\langle \sum a_n z^n B(z), \sum a_m z^m B(z) \right\rangle \\ &= \sum \langle a_n z^n B(z), a_n z^n B(z)\rangle = \sum |a_n|^2 = \|f\|^2.\end{aligned}$$

To prove necessity, observe that the monomials (z^n) form an orthonormal set in $\mathscr{C}(z)$. Since an isometry preserves inner products, $\langle z^m B(z), z^n B(z)\rangle = \delta_{mn}$.

PROOF OF LEMMA 2. The conditions imposed on $B(z)$ in Lemma 1 are met by $f(z)$ and $g(z)$. Thus, $f(z)g(z)$ is in $\mathscr{C}(z)$, while $\|fg\| = \|f\| \cdot \|g\| = 1$. If a_n, b_n are the coefficients of f, g respectively, then $f(z)g(z) = \sum b_n z^n f(z)$, $g(z)f(z) = \sum a_n z^n g(z)$ are valid equations in $\mathscr{C}(z)$. Making use of the conditions on $f(z)$ and $g(z)$, together with the fact that multiplication by z^n is isometric, we find:

$$\begin{aligned}\|f(z)g(z)\|^2 &= \langle f(z)g(z), g(z)f(z)\rangle = \left\langle \sum b_m z^m f(z), \sum a_n z^n g(z)\right\rangle \\ &= \sum \langle b_n f(z), a_n g(z)\rangle = \langle f(z), g(z)\rangle \cdot \sum b_n \bar{a}_n \\ &= \langle f(z), g(z)\rangle \cdot \langle g(z), f(z)\rangle.\end{aligned}$$

The result is that

$$|\langle f(z), g(z)\rangle|^2 = 1 = \|f(z)\|^2 \cdot \|g(z)\|^2,$$

the Schwarz inequality therewith reducing to an equality, and this implies that $f(z)$ and $g(z)$ are linearly dependent.

Proof of Theorem. Let $\mathscr{M}=\mathscr{M}(B)$, where $(z^n B(z))$ is an orthonormal set in $\mathscr{C}(z)$, and let $f(z)$ belong to $\mathscr{M}(B)$. Then necessity is established (as the reader may easily verify) by computing both $z_j P_i f(z)$ and $P_i z_j f(z)$ and observing that they are equal when $i \neq j$.

Now denote by $\mathscr{N}_k$ the set of series $z_k f(z)$ for $f(z)$ in $\mathscr{M}$ $(k=1, \cdots, r)$. Since multiplication by z_k is isometric, while $\mathscr{M}$ is an ideal, each $\mathscr{N}_k$ is a subspace contained in $\mathscr{M}$. The closed span $\bigvee_k \mathscr{N}_k = \mathscr{N}$ is again a subspace, and $\mathscr{N} \subseteq \mathscr{M}$. To prove that $\mathscr{N} \neq \mathscr{M}$, we first show that $P_i P_j = P_j P_i$ for all $i, j = 1, \cdots, r$. The assertion is trivial if $i=j$. If $i \neq j$, let $f(z) = z_i g(z)$ for some $g(z)$ in $\mathscr{M}$. Then

$$P_i P_j f(z) = P_i P_j z_i g(z) = P_i z_i P_j g(z) = z_i P_j g(z),$$
$$P_j P_i f(z) = P_j P_i z_i g(z) = P_j z_i g(z) = z_i P_j g(z),$$

and similarly for $f(z) = z_j g(z)$. Thus $P_i P_j$ and $P_j P_i$ coincide on $\mathscr{N}_i \vee \mathscr{N}_j$. Since P_i and P_j vanish on the complement of $\mathscr{N}_i \vee \mathscr{N}_j$ in $\mathscr{M}$, we have $P_i P_j = P_j P_i$ for $i, j = 1, \cdots, r$. If $r=2$, this implies [**3**, vol. II, p. 55] that the projection operator $P = P_1 \vee P_2$ of $\mathscr{M}$ onto $\mathscr{N} = \mathscr{N}_1 \vee \mathscr{N}_2$ is given by

$$P = 1 - (1 - P_1)(1 - P_2) = P_1 + P_2 - P_1 P_2. \tag{1}$$

For $r > 2$, it is easy to see that

$$P = 1 - (1 - P_1) \cdots (1 - P_r). \tag{1'}$$

To save writing, we take $r=2$ for the moment.

If $g(z)$ is in $\mathscr{N}$, we then have

$$g(z) = Pg(z) = P_1 g(z) + P_2 g(z) - P_1 P_2 g(z). \tag{2}$$

From the definition of the operator P_k, $P_1 g(z) = z_1 g_1(z)$, $P_2 g(z) = z_2 g_2(z)$ for some series $g_1(z)$, $g_2(z)$ in $\mathscr{M}$. The hypothesis $P_1 z_2 g(z) = z_2 P_1 g(z)$ gives

$$P_1 P_2 g(z) = P_1 z_2 g_2(z) = z_2 P_1 g_2(z) = z_1 z_2 g_{12}(z)$$

for some $g_{12}(z)$ in $\mathscr{M}$. Every $g(z)$ in $\mathscr{N}$ is therefore of the form

$$g(z) = z_1 g_1(z) + z_2 g_2(z) - z_1 z_2 g_{12}(z) \tag{3}$$

where $g_1(z)$, $g_2(z)$ and $g_{12}(z)$ are in $\mathscr{M}$.

To complete the proof that $\mathscr{M} \neq \mathscr{N}$, we suppose the contrary: that any $g(z)$ in $\mathscr{M}$ is in $\mathscr{N}$. The representation (3) therefore holds not only for

$g(z)$ but also for $g_1(z)$, $g_2(z)$ and $g_{12}(z)$. By repeating the argument n times we see that

$$g(z) = \sum p_\alpha(z) g_\alpha(z) \tag{4}$$

where $p_\alpha(z)$ is a monomial of degree $\geqq n$ and $g_\alpha(z) \in \mathscr{M}$ for all α. Since n is arbitrary, every $g(z)$ in $\mathscr{M}$ vanishes identically if $\mathscr{M}=\mathscr{N}$. We conclude that $\mathscr{M} \neq \mathscr{N}$. The reasoning holds for $r=2$. For $r>2$, we merely take (1′) instead of (1) as our starting point.

Let $\mathscr{B}$ denote the orthogonal complement of $\mathscr{N}$ in $\mathscr{M}$. Notice that $\mathscr{B}=\bigcap_k \mathscr{B}_k$, where $\mathscr{B}_k=\mathscr{N}_k^\perp \cap \mathscr{M}$. Having shown that the subspace $\mathscr{N} \neq \mathscr{M}$, we may assert that $\mathscr{B}$ contains a nonzero element. Denote by $B(z)$ an element of $\mathscr{B}$ having unit norm. Since $B(z)$ belongs to each of the subspaces $\mathscr{B}_k$, we have $P_k B(z)=0$, for $k=1, \cdots, r$. We proceed to show that $B(z)$ satisfies the condition of Lemma 1: $(z^n B(z))$ is an orthonormal set in $\mathscr{C}(z)$. It is to begin with clear that each element $z^n B(z)$ of the set has unit norm. To prove orthogonality, it must be shown that $\langle z^m B(z), z^n B(z)\rangle = 0$ whenever $m=m_1 \cdots m_r$ and $n=n_1 \cdots n_r$ are unequal, i.e. whenever $m_k \neq n_k$ for at least one k. If $m_i \neq n_i$, all other indices being equal, we have

$$\langle z^m B(z), z^n B(z)\rangle = \langle z_i^{m_i} B(z), z_i^{n_i} B(z)\rangle = 0,$$

since it is clear that each of the sets $(z_k^n B(z))$ $(k=1, \cdots, r)$ is orthonormal in $\mathscr{C}(z)$. If exactly two indices i, j $(j>i)$ differ, with say $m_i - n_i = \mu > 0$, and $n_j - m_j = \nu > 0$, we have

$$\begin{aligned}\langle z^m B(z), z^n B(z)\rangle &= \langle z_i^\mu B(z), z_j^\nu B(z)\rangle \\ &= \langle P_i z_i^\mu B(z), z_j^\nu B(z)\rangle = \langle z_i^\mu B(z), P_i z_j^\nu B(z)\rangle = 0,\end{aligned}$$

since $P_i z_j^\nu B(z) = z_j P_i z_j^{\nu-1} B(z) = \cdots = z_j^\nu P_i B(z)$ and $P_i B(z)=0$. The extension to the general case is immediate.

Given the fact that $(z^n B(z))$ is an orthonormal set, it next follows from Lemma 1 that $\mathscr{M}(B)$ is the closed span of elements $z^n B(z)$. Since $B(z)$ is in $\mathscr{M}$ and $\mathscr{M}$ is closed, $\mathscr{M}(B) \subseteq \mathscr{M}$. Before proceeding with the proof that $\mathscr{M}(B)=\mathscr{M}$, we may conveniently note that $\mathscr{B}$ is of dimension 1. For if $A(z)$ and $B(z)$ are two elements of $\mathscr{B}$ having unit norm, the reasoning of the last paragraph shows that: (i) $(z^n A(z))$ and $(z^n B(z))$ are orthonormal sets in $\mathscr{C}(z)$; (ii) $\langle z^m A(z), z^n B(z)\rangle = 0$ whenever $m \neq n$. It follows from Lemma 2 that $A(z)$ and $B(z)$ are linearly dependent in $\mathscr{C}(z)$: $A(z)=cB(z)$, where c is a complex constant of absolute value 1.

To complete the proof, we show that $\mathscr{L}$, the orthogonal complement of $\mathscr{M}(B)$ in $\mathscr{M}$, has dimension 0. Notice that $\mathscr{B} \subseteq \mathscr{M}(B)$, $\mathscr{M} \cap \mathscr{B}^\perp = \mathscr{N} = \bigvee_k \mathscr{N}_k$ and that

$$\mathscr{L} = \mathscr{M}(B)^\perp \cap \mathscr{M} \subseteq \mathscr{B}^\perp \cap \mathscr{M} = \bigvee_k \mathscr{N}_k.$$

We begin by proving that $\mathscr{L}$ is invariant under each P_j. For this it is enough to show that if $h(z)$ is in $\mathscr{M}(B)$, so is $P_jh(z)$. By linearity and continuity we may assume that $h(z)=z^nB(z)$ for some multi-index n. If $n_j>0$, then $P_jz^nB(z)=z^nB(z)$. If $n_j=0$, the commutativity hypothesis gives $P_jz^nB(z)=z^nP_jB(z)=0$.

Now let $f(z)$ belong to $\mathscr{L}$. Since $\mathscr{L}\subseteq\mathscr{N}$, the reasoning which led to equations (2), (3) above gives

$$f(z) = Pf(z) = P_1f(z) + P_2f(z) - P_1P_2f(z) \tag{2.1}$$

and

$$f(z) = z_1 f_1(z) + z_2f_2(z) - z_1z_2f_{12}(z) \tag{3.1}$$

where $f_1(z)$, $f_2(z)$, $f_{12}(z)$ are in $\mathscr{M}$. By the result of the last paragraph, $P_1f(z)=z_1f_1(z)$ belongs to $\mathscr{L}$. Thus for any $h(z)$ in $\mathscr{M}(B)$,

$$\langle f_1(z), h(z)\rangle = \langle z_1f_1(z), z_1h(z)\rangle = \langle P_1f(z), z_1h(z)\rangle = 0$$

since $\mathscr{M}(B)$ is an ideal. Therefore $f_1(z)$ is in $\mathscr{L}$. Similarly $f_2(z)$ and $f_{12}(z)$ belong to $\mathscr{L}$. The representation (3.1) holds, where $f_1(z), f_2(z), f_{12}(z)$ belong to $\mathscr{L}$. Since $f_1(z), f_2(z)$ and $f_{12}(z)$ have representations of the type (3.1), we can repeat the argument. After n repetitions we have (cf. (4))

$$f(z) = \sum q_\alpha(z)f_\alpha(z) \tag{4.1}$$

where $q_\alpha(z)$ is a monomial of degree $\geqq n$ and $f_\alpha(z)\in\mathscr{L}$ for all α. Therefore $f(z)$ vanishes identically, and $\mathscr{L}=(0)$ as claimed.

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