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SHORTER NOTES

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ON THE INVERSE FUNCTION THEOREM: A COUNTEREXAMPLE

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ABSTRACT. The inverse function theorem for analytic functions has a generalization to $C(X)$ which fails in the disk algebra.

Let A be a commutative complex Banach algebra with identity and let D be an open connected subset of A. A mapping $\Phi: D\rightarrow A$ is analytic in the sense of Lorch (see [3]) if near each point $a \in D$ it has a power series expansion $\Phi(\alpha)=\sum a_n(\alpha-a)^n$. As in classical function theory $a_n=\Phi^{(n)}(a)/n!$. It is known [2] that Φ has a local analytic inverse mapping $\Phi(a)$ to a precisely when $\Phi'(a)$ is an invertible element of A. Classically, when A is the complex numbers, one can say that Φ has a global analytic inverse if Φ is simply one to one on D. This is not true in general. For example take $A=D=C([0, 1])$ and $\Phi[f](x)=xf(x)$. Since $\Phi'[f](x)=x$, the derivative $\Phi'[f]$ is never invertible.

B. W. Glickfeld has shown [2] that the above example is typical when $A=C(X)$ for some compact Hausdorff space X. His theorem is that if $\Phi: D\rightarrow C(X)$ is analytic and one to one, then either the inverse of Φ is analytic or for some $x_0 \in X$ the value $\Phi[f](x_0)$ is constant over all $f\in D$. Glickfeld asks if this result generalizes to other algebras. If we assume X is to be replaced by the maximal ideal space, then in the disk algebra there is a simple counterexample.

Let B be the open unit ball in the disk algebra. Define Φ on B by $\Phi[f]=(f-s)^2$ where $s(z)=z$. If $\Phi[f]=\Phi[g]$, then for each z in the closed unit disk we have either $f(z)=g(z)$ or $f(z)=2z-g(z)$. Since f and g are holomorphic in the open unit disk, the permanence of functional relations [1] implies that either $f=g$ or $f=2s-g$. The second equality cannot hold if f and g both belong to B. For if $g \in B$, then by definition of B the norm of g is less

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than one. From $f=2s-g$ we get $\|f\|\geqq 2\|s\|-\|g\|=2-\|g\|>1$, that is, $f\notin B$. This shows Φ is one to one on B.

Clearly Φ is analytic and $\Phi'[f]=2(f-s)$. Any function $f\in B$ maps the closed unit disk continuously into itself; by the fixed point theorem there is a point z_f with $f(z_f)=z_f$. This gives us $\Phi'[f](z_f)=2(f(z_f)-z_f)=0$. In other words $\Phi'[f]$ is never invertible, and Φ cannot have an analytic inverse.

Let z_0 be a point in the closed unit disk, which is the maximal ideal space of the disk algebra. In contrast to Glickfeld's theorem the set $\{\Phi[f](z_0); f\in B\}$ always contains more than one point; indeed it is an open subset of the plane.

References

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