ON THE HOMOTOPY TYPE OF IRREGULAR SETS
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Abstract. If M is an open connected manifold and h is a homeomorphism of M onto itself such that h is positively regular on M and the set of irregular points, Irr(h), is a nonseparating compactum, then it is shown that Irr(h) is a strong deformation retract of M.

1. Introduction. If (X, d) is a metric space and h is a homeomorphism (continuous map) of X into itself, h is regular (positively regular) at p ∈ X if, for each ε > 0, there exists δ > 0 such that d(p, q) < δ implies that d(h^n(p), h^n(q)) < ε for all integers n (positive integers n). Let Irr(h) denote the set of points at which h fails to be regular. If X is an open connected manifold, h is a homeomorphism of X onto itself which is positively regular at each point of X, and if Irr(h) is a compact zero dimensional nonempty subset of X, it follows from [6] that Irr(h) contains a single point and X is homeomorphic to Euclidean n-space.

If P is a compact polyhedron in Euclidean space and X is an open regular neighborhood of P, the homeomorphism which pushes along the mapping cylinder structure toward P is positively regular on X and Irr(h) = P. In [1] and [2], we investigated the problem of the converse of this construction. However in [1] and [2], we limited our considerations to the case when h|Irr(h) is periodic and were able to show that Irr(h) is a strong deformation retract of X. In this note, we remove the condition that h|Irr(h) is periodic.

2. A retraction theorem. Let M be a locally compact metric space and let f: M → M be a map which is positively regular at each point. For x ∈ M, let O(x) = cl{f^n(x)} for i = 1, ..., 0 and let K(x) = ∩ i=0 O(f^n(x)). Given y, z ∈ O(x), define y • z = lim i→+∞ f^i(y) for z ∈ K(x). We summarize some facts from [3].

Theorem 1. If O(x) is compact, then the product described above is well-defined and O(x) with this product is a commutative topological semigroup. K(x) is a topological subgroup of O(x). If y ∈ K(x), then K(y) = O(y) = K(x).

With notation as above, let C = ∪ x∈M K(x). For each x ∈ M, let e_x be the identity element in K(x) with respect to the product structure from

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O(x). [Note that even though $K(x)$ might be equal to $K(y)$ for some $x \neq y$, their product structures might be different.] If $x \in C$, it follows from Theorem 1 that $x \in K(x)$ and since $x = \lim_{t \to +\infty} f^t(x), x = e_x$. Define a function $r: M \to C$ by $r(x) = e_x$.

**Theorem 2.** If, for each $x \in M$, $O(x)$ is compact, then $r$ is a retraction of $M$ onto $C$.

**Proof.** By the remarks above, $r|C$ is the identity map. We proceed to show that $r$ is continuous at $x \in M$. For $y \in M$, let

$$A(y) = \left\{ p \in K(y) \mid p = \lim_{t \to +\infty} f^t_n(y) \text{ where } \lim_{t \to +\infty} f^t_n(x) = e_x \right\}.$$ 

Choose any increasing sequence \( \{n_i\}_{i=1}^{\infty} \) such that $e_x = \lim_{i \to +\infty} f^{n_i}(x)$. Since $O(y)$ is compact, \( \{f^{n_i}(y)\} \) has a subsequence which converges to a point in $K(y)$; hence $A(y)$ is nonempty. We claim that $A(y)$ is closed; suppose that $p = \lim_{i \to +\infty} p_i$, where $p_i \in A(y)$, $\rho_i = \lim_{j \to +\infty} f^{n_i(j)}(y)$ and $\rho_j = \lim_{j \to +\infty} f^{n_j(i)}(x) = e_x$. For each $i$, choose $j_i$ such that $d(f^{n_i(j_i)}(x), e_x) < 1/i$ and $d(f^{n_j(j_i)}(y), p_i) < 1/i$. Therefore $\lim_{i \to +\infty} f^{n_i(j_i)}(x) = e_x$ and $\lim_{i \to +\infty} f^{n_i(j_i)}(y) = p$ so that $p \in A(y)$ and $A(y)$ is closed.

Suppose $p = \lim_{i \to +\infty} f^{n_i}(y)$ and $q = \lim_{i \to +\infty} f^{m_i}(y)$ where $e_x = \lim_{i \to +\infty} f^{n_i}(x) = \lim_{i \to +\infty} f^{m_i}(x)$. Since $e_x = e_y \cdot e_x = \lim_{i \to +\infty} f^{n_i+m_i}(x), p \cdot q \in A(y)$. Therefore $A(y)$ is a subsemigroup of $K(y)$. Since $A(y)$ is compact, $A(y)$ contains an idempotent [7, p. 22] which must be $e_y$.

Let $\varepsilon > 0$ be given and let $\delta > 0$ be such that $d(x, y) < \delta$ implies

$$d(f^t(x), f^t(y)) < \varepsilon/3$$

for all $t > 0$. Suppose $d(x, y) < \delta$. Since $e_y \in A(y)$, there exists a sequence $\{n_i\}_{i=1}^{\infty}$ such that $\lim_{i \to +\infty} f^{n_i}(x) = e_x$ and $\lim_{i \to +\infty} f^{n_i}(y) = e_y$. Then

$$d(r(x), r(y)) = d(e_x, e_y) \leq d(e_x, f^{n_i}(x)) + d(f^{n_i}(x), f^{n_i}(y)) + d(f^{n_i}(y), e_y)$$

and by choosing $i$ sufficiently large, we have that $d(r(x), r(y)) < \varepsilon$. Hence $r$ is continuous at $x$.

3. Irregular sets. Now let $M$ be an open connected manifold, $Y \subseteq M$ a compact set which does not separate $M$ and let $h$ be a homeomorphism of $M$ onto itself such that $h$ is positively regular at each point and $\text{Irr}(h) = Y$. Assume that the metric of $M$ is induced from the metric of the one point compactification of $M$. From the techniques of Proposition 2.1 of [2], $Y$ is connected.

**Lemma 3.** If $A \subseteq M$ is a compact set and $U$ is a neighborhood of $Y$, then there is an integer $N$ such that $h^n(A) \subseteq U$ for all $n > N$. 

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Proof. This was shown in Corollary 2.2 of [2] with the additional hypothesis that the extension, \( h_\infty \), of \( h \) to the one point compactification, \( M_\infty = M \cup \{ \infty \} \), of \( M \) is not regular at \( \infty \). However, if \( h_\infty \) were regular at \( \infty \), then \( h_\infty \) would be positively regular on the compactum \( M_\infty \). But then \( h_\infty \) would be regular on all \( M_\infty \) and \( \text{Irr}(h) = \emptyset \) [5].

Lemma 4. \( Y = \bigcup_{x \in M} K(x) \).

Proof. Let \( Z = \bigcup_{x \in M} K(x) \). It follows from Lemma 3 that \( Z \subseteq Y \) and \( Z \) is nonempty. We claim that \( Z \) is closed; suppose that \( p = \lim_{i \to +\infty} p_i \) where \( p_i \in K(x_i) \). Let \( W \) be a compact neighborhood of \( Y \). By Lemma 3, we may assume that \( x_i \in W \) for each \( i \), and thus we may assume that there is a point \( x \in W \) such that \( x = \lim_{i \to +\infty} x_i \). Let \( \varepsilon > 0 \) be given and let \( \delta > 0 \) be such that if \( d(y, x) < \delta \); then \( d(h^n(y), h^n(x)) < \varepsilon/3 \) for each \( n > 0 \). Choose \( N > 0 \) so that \( d(x_N, x) < \delta \) and \( d(p_N, p) < \varepsilon/3 \). Now there exists \( n > 0 \) such that \( d(h^n(x_N), p_N) < \varepsilon/3 \); thus \( d(h^n(y), p) \leq d(h^n(x), h^n(x_N)) + d(h^n(x_N), p_N) + d(p_N, p) < \varepsilon \). It follows that \( p \in K(x) \); thus \( Z \) is closed.

For each open set \( U \) containing \( Z \) and any compact set \( A \subseteq M \), we claim that \( h^n(A) \subseteq U \) for all but finitely many positive integers \( n \). Clearly this is true if \( A \) is a point; the general statement follows from positive regularity and standard compactness arguments.

Suppose \( q \in Y - Z \); let \( W \) be a compact connected neighborhood of \( Y \) and let \( U \) be an open neighborhood of \( Z \) such that \( q \notin U \). For some \( n > 0 \), \( h^n(W) \subseteq U \) and since \( Y \) is connected, \( h^n(\text{frontier } W) \cap Y = \emptyset \). Since \( M - Y \) and \( Y \) are invariant under \( h \), this is a contradiction. Thus \( Y = Z \).

Theorem 5. \( Y \) is a strong deformation retract of \( M \).

Proof. By Lemma 4 and Theorem 2, \( Y \) is a retract of \( M \); therefore \( Y \) is an ANR. To prove the theorem it suffices to show that \( j_* : \pi_n(Y) \to \pi_n(M) \) is an isomorphism for each \( n \), where \( j : Y \to M \) is the inclusion map [4, p. 218]. Since \( Y \) is a retract of \( M \), \( j_* \) is one-to-one and since \( Y \) is an ANR, there is a neighborhood \( U \) of \( Y \) and a retraction \( \rho : U \to Y \) such that the diagram

\[
\begin{array}{ccc}
U & \xrightarrow{k} & M \\
\downarrow{\rho} & & \\
Y & \xrightarrow{i} & Y
\end{array}
\]

is homotopy commutative (rel \( Y \)) in \( M \), where all maps other than \( \rho \) are inclusions. Let \( y \in Y \), let \( \alpha \in \pi_n(M, y) \) and let \( f : (S^n, *) \to (M, y) \) be a map which represents \( \alpha \). For some \( m > 0 \), \( h^m(f(S^n)) \subseteq U \); let \( \beta \) denote the class of
Remark. Note that Theorem 5 is true in the case that $M$ is a finite dimensional ANR.

Corollary 6. Let $M$ be a connected open manifold and let $h$ be a homeomorphism of $M$ onto itself such that $h$ is positively regular at each point. If $\text{Irr}(h)$ is a continuum in $M$ and if for some $x \in M$, $\lim_{t \to +\infty} \sup \ell(t) = \text{Irr}(h)$, then $\text{Irr}(h)$ is the product of 1-spheres.

Proof. It is easily seen that $K(x) = \text{Irr}(h)$ and hence by Theorem 1, $\text{Irr}(h)$ is a commutative topological group. By either Theorem 2 or 5, $\text{Irr}(h)$ is locally connected and the result follows from [8, p. 262].

Remark. In [2], we gave an example which can be slightly modified to an example for which $M$ is the product of the 1-sphere and 3-dimensional Euclidean space, $\text{Irr}(h)$ is a wildly embedded one sphere in $M$ and for each $x \in M$, $\lim_{t \to +\infty} \sup \ell(t) = \text{Irr}(h)$. If $r : M \to \text{Irr}(h)$ is the retraction defined in §2, then for each $x \in \text{Irr}(h)$, $r^{-1}(x)$ is a generalized cohomology 3-manifold which is not a 3-manifold.

Conjecture. If $M$ and $h$ are as in Corollary 6 and if $r : M \to \text{Irr}(h)$ is the retraction defined in §2, then, for each $x \in \text{Irr}(h)$, $r^{-1}(x)$ is a generalized cohomology manifold.

A positive answer to the conjecture would provide valuable information about the topological conjugacy class of $h$.

References

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