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ON THE HOMOTOPY TYPE OF IRREGULAR SETS

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ABSTRACT. If M is an open connected manifold and h is a homeomorphism of M onto itself such that h is positively regular on M and the set of irregular points, $\mathrm{Irr}(h)$, is a nonseparating compactum, then it is shown that $\mathrm{Irr}(h)$ is a strong deformation retract of M.

1. **Introduction.** If (X, d) is a metric space and h is a homeomorphism (continuous map) of X into itself, h is *regular* (*positively regular*) at $p \in X$ if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $d(p, q) < \delta$ implies that $d(h^n(p), h^n(q)) < \varepsilon$ for all integers n (positive integers n). Let $\mathrm{Irr}(h)$ denote the set of points at which h fails to be regular. If X is an open connected manifold, h is a homeomorphism of X onto itself which is positively regular at each point of X, and if $\mathrm{Irr}(h)$ is a compact zero dimensional nonempty subset of X, it follows from [6] that $\mathrm{Irr}(h)$ contains a single point and X is homeomorphic to Euclidean n-space.

If P is a compact polyhedron in Euclidean space and X is an open regular neighborhood of P, the homeomorphism which pushes along the mapping cylinder structure toward P is positively regular on X and $\mathrm{Irr}(h) = P$. In [1] and [2], we investigated the problem of the converse of this construction. However in [1] and [2], we limited our considerations to the case when $h|\mathrm{Irr}(h)$ is periodic and were able to show that $\mathrm{Irr}(h)$ is a strong deformation retract of X. In this note, we remove the condition that $h|\mathrm{Irr}(h)$ is periodic.

2. **A retraction theorem.** Let M be a locally compact metric space and let $f: M \to M$ be a map which is positively regular at each point. For $x \in M$, let $O(x) = \mathrm{cl}\{f^i(x)\}_{i=1}^{\infty}$ and let $K(x) = \bigcap_{i=1}^{\infty} O(f^i(x))$. Given $y, z \in O(x)$, define $y \cdot z = \lim_{i \to +\infty} f^{m_i + n_i}(x)$ where $y = \lim_{i \to +\infty} f^{m_i}(x)$ and $z = \lim_{i \to +\infty} f^{n_i}(x)$. We summarize some facts from [3].

THEOREM 1. *If $O(x)$ is compact, then the product described above is well-defined and $O(x)$ with this product is a commutative topological semigroup. $K(x)$ is a topological subgroup of $O(x)$. If $y \in K(x)$, then $K(y) = O(y) = K(x)$.*

With notation as above, let $C = \bigcup_{x \in M} K(x)$. For each $x \in M$, let e_x be the identity element in $K(x)$ with respect to the product structure from

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$O(x)$. [Note that even though $K(x)$ might be equal to $K(y)$ for some $x\neq y$, their product structures might be different.] If $x\in C$, it follows from Theorem 1 that $x\in K(x)$ and since $x=\lim_{i\to+\infty} f^0(x)$, $x=e_x$. Define a function $r:M\to C$ by $r(x)=e_x$.

THEOREM 2. *If, for each $x\in M$, $O(x)$ is compact, then r is a retraction of M onto C.*

PROOF. By the remarks above, $r|C$ is the identity map. We proceed to show that r is continuous at $x\in M$. For $y\in M$, let

$$A(y)=\left\{p\in K(y)\ \middle|\ p=\lim_{i\to+\infty} f^{n_i}(y) \text{ where } \lim_{i\to+\infty} f^{n_i}(x)=e_x\right\}.$$

Choose any increasing sequence $\{n_i\}_{i=1}^{\infty}$ such that $e_x=\lim_{i\to+\infty} f^{n_i}(x)$. Since $O(y)$ is compact, $\{f^{n_i}(y)\}$ has a subsequence which converges to a point in $K(y)$; hence $A(y)$ is nonempty. We claim that $A(y)$ is closed; suppose that $p=\lim_{i\to+\infty} p_i$, where $p_i\in A(y)$, $p_i=\lim_{j\to+\infty} f^{n(i,j)}(y)$ and $\lim_{j\to+\infty} f^{n(i,j)}(x)=e_x$. For each i, choose j_i such that $d(f^{n(i,j_i)}(x), e_x)<1/i$ and $d(f^{n(i,j_i)}(y), p_i)<1/i$. Therefore $\lim_{i\to+\infty} f^{n(i,j_i)}(x)=e_x$ and $\lim_{i\to+\infty} f^{n(i,j_i)}(y)=p$ so that $p\in A(y)$ and $A(y)$ is closed.

Suppose $p=\lim_{i\to+\infty} f^{n_i}(y)$ and $q=\lim_{i\to+\infty} f^{m_i}(y)$ where $e_x=\lim_{i\to+\infty} f^{n_i}(x)=\lim_{i\to+\infty} f^{m_i}(x)$. Since $e_x=e_x\cdot e_x=\lim_{i\to+\infty} f^{n_i+m_i}(x)$, $p\cdot q\in A(y)$. Therefore $A(y)$ is a subsemigroup of $K(y)$. Since $A(y)$ is compact, $A(y)$ contains an idempotent [7, p. 22] which must be e_y.

Let $\varepsilon>0$ be given and let $\delta>0$ be such that $d(x,y)<\delta$ implies

$$d(f^i(x), f^i(y)) < \varepsilon/3$$

for all $i>0$. Suppose $d(x,y)<\delta$. Since $e_y\in A(y)$, there exists a sequence $\{n_i\}_{i=1}^{\infty}$ such that $\lim_{i\to+\infty} f^{n_i}(x)=e_x$ and $\lim_{i\to+\infty} f^{n_i}(y)=e_y$. Then

$$\begin{aligned} d(r(x), r(y)) &= d(e_x, e_y) \\ &\leqq d(e_x, f^{n_i}(x)) + d(f^{n_i}(x), f^{n_i}(y)) + d(f^{n_i}(y), e_y) \end{aligned}$$

and by choosing i sufficiently large, we have that $d(r(x), r(y))<\varepsilon$. Hence r is continuous at x.

3. **Irregular sets.** Now let M be an open connected manifold, $Y\subseteq M$ a compact set which does not separate M and let h be a homeomorphism of M onto itself such that h is positively regular at each point and $\mathrm{Irr}(h)=Y$. Assume that the metric of M is induced from the metric of the one point compactification of M. From the techniques of Proposition 2.1 of [**2**], Y is connected.

LEMMA 3. *If $A\subseteq M$ is a compact set and U is a neighborhood of Y, then there is an integer N such that $h^n(A)\subseteq U$ for all $n>N$.*

PROOF. This was shown in Corollary 2.2 of [2] with the additional hypothesis that the extension, h_∞, of h to the one point compactification, $M_\infty = M \cup \{\infty\}$, of M is not regular at ∞. However, if h_∞ were regular at ∞, then h_∞ would be positively regular on the compactum M_∞. But then h_∞ would be regular on all M_∞ and $\mathrm{Irr}(h) = \varnothing$ [5].

LEMMA 4. $Y = \bigcup_{x \in M} K(x)$.

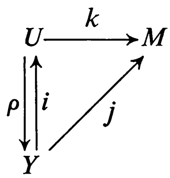
PROOF. Let $Z = \bigcup_{x \in M} K(x)$. It follows from Lemma 3 that $Z \subseteq Y$ and Z is nonempty. We claim that Z is closed; suppose that $p = \lim_{i \to +\infty} p_i$ where $p_i \in K(x_i)$. Let W be a compact neighborhood of Y. By Lemma 3, we may assume that $x_i \in W$ for each i, and thus we may assume that there is a point $x \in W$ such that $x = \lim_{i \to +\infty} x_i$. Let $\varepsilon > 0$ be given and let $\delta > 0$ be such that if $d(y, x) < \delta$; then $d(h^n(y), h^n(x)) < \varepsilon/3$ for each $n > 0$. Choose $N > 0$ so that $d(x_N, x) < \delta$ and $d(p_N, p) < \varepsilon/3$. Now there exists $n > 0$ such that $d(h^n(x_N), p_N) < \varepsilon/3$; thus $d(h^n(x), p) \leqq d(h^n(x), h^n(x_N)) + d(h^n(x_N), p_N) + d(p_N, p) < \varepsilon$. It follows that $p \in K(x)$; thus Z is closed.

For each open set U containing Z and any compact set $A \subseteq M$, we claim that $h^n(A) \subseteq U$ for all but finitely many positive integers n. Clearly this is true if A is a point; the general statement follows from positive regularity and standard compactness arguments.

Suppose $q \in Y - Z$; let W be a compact connected neighborhood of Y and let U be an open neighborhood of Z such that $q \notin U$. For some $n > 0$, $h^n(W) \subseteq U$ and since Y is connected, h^n (frontier W) $\cap Y \neq \varnothing$. Since $M - Y$ and Y are invariant under h, this is a contradiction. Thus $Y = Z$.

THEOREM 5. *Y is a strong deformation retract of M.*

PROOF. By Lemma 4 and Theorem 2, Y is a retract of M; therefore Y is an ANR. To prove the theorem it suffices to show that $j_*: \pi_n(Y) \to \pi_n(M)$ is an isomorphism for each n, where $j: Y \to M$ is the inclusion map [4, p. 218]. Since Y is a retract of M, j_* is one-to-one and since Y is an ANR, there is a neighborhood U of Y and a retraction $\rho: U \to Y$ such that the diagram



is homotopy commutative (rel Y) in M, where all maps other than ρ are inclusions. Let $y \in Y$, let $\alpha \in \pi_n(M, y)$ and let $f: (S^n, *) \to (M, y)$ be a map which represents α. For some $m > 0$, $h^m(f(S^n)) \subseteq U$; let β denote the class of

$h^m f$ in $\pi_n(U, h^m(y))$. Then $j_*[(h|Y)_*^{-m}\rho_*(\beta)] = h_*^{-m} j_* \rho_*(\beta) = h_*^{-m} k_*(\beta) = h_*^{-m} h_*^m(\alpha) = \alpha$.

REMARK. Note that Theorem 5 is true in the case that M is a finite dimensional ANR.

COROLLARY 6. *Let M be a connected open manifold and let h be a homeomorphism of M onto itself such that h is positively regular at each point. If* $\mathrm{Irr}(h)$ *is a continuum in M and if for some $x \in M$,* $\lim_{i\to+\infty} \sup h^i(x) = \mathrm{Irr}(h)$, *then* $\mathrm{Irr}(h)$ *is the product of* 1*-spheres.*

PROOF. It is easily seen that $K(x) = \mathrm{Irr}(h)$ and hence by Theorem 1, $\mathrm{Irr}(h)$ is a commutative topological group. By either Theorem 2 or 5, $\mathrm{Irr}(h)$ is locally connected and the result follows from [**8**, p. 262].

REMARK. In [**2**], we gave an example which can be slightly modified to an example for which M is the product of the 1-sphere and 3-dimensional Euclidean space, $\mathrm{Irr}(h)$ is a wildly embedded one sphere in M and for each $x \in M$, $\lim_{i\to+\infty} \sup h^i(x) = \mathrm{Irr}(h)$. If $r: M \to \mathrm{Irr}(h)$ is the retraction defined in §2, then for each $x \in \mathrm{Irr}(h)$, $r^{-1}(x)$ is a generalized cohomology 3-manifold which is not a 3-manifold.

CONJECTURE. *If M and h are as in Corollary* 6 *and if* $r: M \to \mathrm{Irr}(h)$ *is the retraction defined in* §2, *then, for each* $x \in \mathrm{Irr}(h)$, $r^{-1}(x)$ *is a generalized cohomology manifold.*

A positive answer to the conjecture would provide valuable information about the topological conjugacy class of h.

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