PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 38, Number 2, April 1973

HURWITZ' THEOREM

J. H. E. COHN

ABSTRACT. If $[a_0, a_1, a_2, \cdots]$ is the continued fraction for a real number x, and p_n/q_n the nth convergent, define $\theta_n = q_n|p_n - xq_n|$. Hurwitz' Theorem asserts that $\phi_n = \min\{\theta_{n-1}, \theta_n, \theta_{n+1}\} < 5^{-1/2}$ whenever ϕ_n is defined. It is the object of this note to provide a simple proof of this fact.

From the well-known relation

$$\left|x - \frac{p_n}{q_n}\right| + \left|x - \frac{p_{n+1}}{q_{n+1}}\right| = \frac{1}{q_n q_{n+1}}$$

we obtain

$$\left(\frac{q_{n+1}}{q_n}\right)^2 \theta_n - \left(\frac{q_{n+1}}{q_n}\right) + \theta_{n+1} = 0 \tag{1}$$

and so

$$\frac{q_{n+1}}{q_n} = \frac{1 \pm (1 - 4\theta_n\theta_{n+1})^{1/2}}{2\theta_n},$$

with a similar result when n is replaced by $n-1$ throughout. Thus

$$\begin{aligned} \frac{1 + (1 - 4\theta_n\theta_{n+1})^{1/2}}{2\theta_n} \geqq \frac{q_{n+1}}{q_n} &= a_{n+1} + \frac{q_{n-1}}{q_n} \\ &\geqq 1 + \frac{2\theta_{n-1}}{1 + (1 - 4\theta_n\theta_{n-1})^{1/2}} \\ &= 1 + \frac{1 - (1 - 4\theta_n\theta_{n-1})^{1/2}}{2\theta_n}, \end{aligned}$$

or

$$2\theta_n \leqq (1 - 4\theta_n\theta_{n+1})^{1/2} + (1 - 4\theta_n\theta_{n-1})^{1/2}.$$

Then $2\phi_n \leqq 2(1 - 4\phi_n^2)^{1/2}$, whence $\phi_n \leqq 5^{-1/2}$. But equality would require that $\theta_n = \theta_{n-1} = \theta_{n+1} = 5^{-1/2}$, impossible in view of (1).

DEPARTMENT OF MATHEMATICS, ROYAL HOLLOWAY COLLEGE, ENGLEFIELD GREEN, SURREY, ENGLAND

Received by the editors October 12, 1972.
AMS (MOS) subject classifications (1970). Primary 10F20.
Key words and phrases. Hurwitz Theorem, continued fraction approximation.