

ON CONVEX POWER SERIES OF A CONSERVATIVE MARKOV OPERATOR

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ABSTRACT. A. Brunel proved that a conservative Markov operator, P , has a finite invariant measure if and only if every operator $Q = \sum_{n=0}^{\infty} \alpha_n P^n$ where $\alpha_n \geq 0$ and $\sum \alpha_n = 1$ is conservative.

In this note we give a different proof and study related problems.

Introduction. We shall use the notation and definitions of [3]. Let us quote some basic results:

The operator P is conservative if and only if for every $0 \leq f \in L_\infty$ the sum $\sum_{n=0}^{\infty} P^n f$ assumes the values 0 or ∞ only.

The operator P is conservative if and only if whenever $0 \leq f \in L_\infty$ and $Pf \leq f$ then $Pf = f$. See [4, Corollary 1].

If P is conservative and $Pf = f$, then f is $\Sigma_i(P)$ measurable where $\Sigma_i(P) = \{A : P1_A = 1_A\}$. See [3, Theorem A of Chapter III].

We shall study operators of the form $Q = \sum_{n=0}^{\infty} \alpha_n P^n$ where $\alpha_n \geq 0$ and $\sum \alpha_n = 1$. Such operators will be called convex power series of P , and denoted by $A(P)$ where $A(z) = \sum_{n=0}^{\infty} \alpha_n z^n$.

1. Conditions for Q to be conservative.

THEOREM 1.1. *Let P be a conservative operator and $Q = A(P)$ a convex power series of P . If $\sum_{n=1}^{\infty} n\alpha_n < \infty$ then Q is conservative too.*

PROOF. Note first that

$$\sum_{n=0}^{\infty} (1 - \alpha_0 - \cdots - \alpha_n) = \sum_{n=0}^{\infty} \left(\sum_{k=n+1}^{\infty} \alpha_k \right) = \sum_{n=1}^{\infty} n\alpha_n.$$

Put $\gamma_n = 1 - (\alpha_0 + \cdots + \alpha_n)$ then, by assumption, $\sum \gamma_n < \infty$. Define $K = \sum \gamma_n P^n$ then K is a positive bounded operator on L_1 . An easy computation shows that $I - Q = (I - P)K$. Thus if $0 \leq f \in L_\infty$ and $(I - Q)f \geq 0$ then $(I - P)Kf \geq 0$ and $(I - P)Kf = 0$ because P is conservative. By the characterization of conservative operators, given in the introduction, Q is conservative too.

REMARK. Every finite convex combination of P^k is conservative.

Received by the editors December 17, 1971.

AMS (MOS) subject classifications (1970). Primary 28A65.

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For the rest of this section we shall assume that P is ergodic and conservative and study $\Sigma_i(Q)$. If $A \in \Sigma_i(Q)$ or $1_A = \sum \alpha_n P^n 1_A$ then whenever $\alpha_n \neq 0$ $P^n 1_A \leq 1_A$ (since $P^n 1_A \leq 1$) and thus $P^n 1_A = 1_A$ since P^n is conservative.

Let r be the maximal common divisor of n such that $\alpha_n \neq 0$. Then, on the one hand, $Q = \sum_{n=0}^\infty \alpha_{nr} P^{nr}$, and on the other hand there exist $n_1 \cdots n_j$ with $\alpha_{n_k} \neq 0$ and $nr = q_1 n_1 + \cdots + q_j n_j$ with q_i positive integers and $n \geq N$. Thus, $1_A = P^{(N+1)r} 1_A = P^r P^N 1_A = P^r 1_A$. To summarize $\Sigma_i(Q) = \Sigma_i(P^r)$ for some integer r .

LEMMA 1.2. *Let P be ergodic and conservative then $\Sigma_i(P^r)$ is atomic.*

PROOF. Let us assume, to the contrary, that $A_n \in \Sigma_i(P^r)$ where $A_n \downarrow \emptyset$. Now $0 = (I - P^r) 1_{A_n} = (I - P)(I + P + \cdots + P^{r-1}) 1_{A_n}$. Since P is ergodic and conservative $(I + P + \cdots + P^{r-1}) 1_{A_n}$ is a constant but $1_{A_n}(x) = 1$ if $x \in A_n$. Thus $(I + P + \cdots + P^{r-1}) 1_{A_n} \geq 1$ and this contradicts the continuity of P on L_1 .

Let us take an atom A , of $\Sigma_i(P^r)$. Put $P^j 1_A = f$, $1 \leq j \leq r - 1$. Note that $f \in \Sigma_i(P^r)$ too. Put $B_\varepsilon = \{x : f(x) \geq \varepsilon\}$, $B_0 = \{x : f(x) > 0\}$; both sets are in $\Sigma_i(P^r)$. Now $f \geq \varepsilon 1_{B_\varepsilon}$ thus $1_A = P^{r-j} f \geq \varepsilon P^{r-j} 1_{B_\varepsilon}$ hence $P^{r-j} 1_{B_\varepsilon} \leq 1_A$. Let $\varepsilon \rightarrow 0$ to conclude $P^{r-j} 1_B \leq 1_A$. Since A is an atom and $P^{r-j} 1_B$ is invariant under P^r , we must have $P^{r-j} 1_B = \text{const } 1_A$. Now $P^r 1_B = 1_B$ so the constant is one. Thus $P^{r-j} 1_B = P^{r-j} f$ but $1_B \geq f$. Therefore, $\sum P^n (1_B - f) < \infty$ or $f = 1_B$.

THEOREM 1.3. $\Sigma_i(P^r) = \{A_0, \dots, A_{k-1}\}$ where the sets A_i are disjoint, k divides r and $1_{A_1} = P 1_{A_0}$, $1_{A_2} = P 1_{A_1}$, \dots , $1_{A_0} = P 1_{A_{k-1}}$.

PROOF. Let A_0 be an atom of $\Sigma_i(P^r)$. By the above argument $P^j 1_{A_0} = 1_A$ and $A_j \in \Sigma_i(P^r)$. Let k be the first integer such that $P^k 1_{A_0} = 1_{A_0}$. Clearly k divides r and the sets A_j , $1 \leq j \leq k - 1$, are disjoint: If $B \subset A_j$ and $B \in \Sigma_i(P^r)$ then $P^{k-j} 1_B \leq 1_{A_0}$ and like the previous argument must be equal to 1_{A_0} or $P^{k-j} 1_B = P^{k-j} 1_{A_j}$ and $B = A_j$.

Now $\bigcup_{j=0}^{k-1} A_j$ is invariant under P and thus must be all of X . Clearly each A_i is an atom of $\Sigma_i(P^r)$ and $\Sigma_i(P^r) = \{A_0, A_1, \dots, A_{k-1}\}$.

REMARK. Note that if n divides m then $\Sigma_i(P^n) \subset \Sigma_i(P^m)$. Thus $\bigvee \Sigma_i(P^n) = \bigvee \Sigma_i(P^{n!})$ and $\Sigma_i(P^{n!})$ is monotone in n .

Theorem 1.3 was proved by Moy in [7, Theorem 1] for a more general case by a different method.

2. Conditions for Q to be dissipative.

LEMMA 2.1. *Let P_1 and P_2 be commuting elements of a Banach algebra with $\|P_1\| = \|P_2\| = 1$. Let $Q = \alpha P_1 + \beta P_2$ where $0 < \alpha, \beta < 1$ and $\alpha + \beta = 1$. Then $\|Q^n (P_1 - P_2)\| \leq (K/\sqrt{n}) \cdot \alpha \cdot \beta$ where K is a constant.*

PROOF. First let us reduce the problem to the case where $\alpha=\beta=\frac{1}{2}$: If $\alpha < \frac{1}{2}$ then $Q = \frac{1}{2}(P'_1 + P_2)$ where $P'_1 = 2\alpha P_1 + (1-2\alpha)P_2$ and $P'_1 - P_2 = 2\alpha(P_1 - P_2)$.

Following [8] we write

$$Q^n(P_1 - P_2) = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} P_1^k P_2^{n-k} (P_1 - P_2)$$

$$= \frac{1}{2^n} \sum_{k=1}^n \left[\binom{n}{k-1} - \binom{n}{k} \right] P_1^k P_2^{n-k+1} + \frac{1}{2^n} P_1^{n+1} - \frac{1}{2^n} P_2^{n+1}$$

thus

$$\|Q^n(P_1 - P_2)\| \leq \frac{1}{2^{n-1}} + \frac{1}{2^n} \sum_{k=1}^n \left| \binom{n}{k-1} - \binom{n}{k} \right|.$$

We may assume that n is even. Since $\binom{n}{k}$ increases as k increases from 0 to $n/2$ and then decreases as k goes from $n/2$ to n the sum of absolute values is bounded by $(2/2^n)\binom{n}{n/2}$ which is, by Stirling's formula, bounded by K/\sqrt{n} , and hence the lemma follows.

Let P be an operator and $Q = \sum_{n=0}^{\infty} \alpha_n P^n$, $\alpha_i \geq 0$, $\sum \alpha_i = 1$. Assume that α_i and α_j ($i < j$) are the first nonzero coefficients. Put $Q' = \sum \alpha_n P^{n-i}$ where $P^i Q' = Q' P^i = Q$. Choose $0 < \gamma < \min\{\alpha_i, \alpha_j, \frac{1}{2}\}$. Then

$$Q' = \gamma(I + P^{j-i}) + \sum \beta_n P^n = \gamma(I + P^{j-i}) + Q_1.$$

Note $\sum \beta_n + 2\gamma = 1$, $\beta_n \geq 0$. Thus $Q' = \frac{1}{2}[(2\gamma + Q_1) + (2\gamma P^{j-i} + Q_1)]$ and $\|Q'^n(I - P^{j-i})\| \rightarrow 0$. Therefore, $\|Q^n(I - P^{j-i})\| = \|P^{in} Q'^n(I - P^{j-i})\| \rightarrow 0$.

THEOREM 2.2. *Let P be an operator with no invariant measure. Let $Q = A(P)$ be a convex power series of P such that $A(z)$ has at least two nonzero coefficients. There exists a set A , $m(A) > 0$, such that $\|Q^n 1_A\|_{\infty} \rightarrow 0$.*

PROOF. Note first that, by a standard argument, P^{j-i} has no invariant measure. By [3, Chapter IV, (4.9)] there exists a set A with $m(A) > 0$, such that if $\lambda P^{j-i} = \lambda$ then $\lambda(A) = 0$ ($\lambda \in L^*_\infty$). By the Hahn-Banach Theorem

$$1_A \in \overline{\text{Range}(I - P^{j-i})}.$$

Thus $\|Q^n 1_A\|_{\infty} \rightarrow 0$.

REMARK. In [1, Lemma 1] Brunel proves that property for $Q = (1/e)\exp P$.

THEOREM 2.3. *Let Q be a Markov operator such that, for some $0 \leq h \in L_\infty$, $\|Q^n h\| \rightarrow 0$. Choose a sequence N_m such that $\sum_{m=0}^{\infty} \|Q^{N_m} h\| < \infty$. The operator $R = \sum \alpha_n Q^n$ is not conservative if $\sum_{m=0}^{\infty} (\sum_{n=0}^{N_m-1} \alpha_n)^m < \infty$.*

PROOF. Put

$$R = R_{1,m} + R_{2,m} = \sum_{n=0}^{N_m-1} \alpha_n Q^n + \sum_{n=N_m}^{\infty} \alpha_n Q^n.$$

Then $R^m = R_{1,m}^m + S_m Q^{N_m}$ where S_m is of the form $\sum \beta_n Q^n$, $\sum \beta_n \leq 1$, $\beta_n \geq 0$. Thus

$$R^m h = R_{1,m}^m h + S_m Q^{N_m} h \leq R_{1,m}^m h + \|Q^{N_m} h\|.$$

Since $\sum_{m=0}^\infty \|Q^{N_m} h\| < \infty$ it is enough to consider the first term:

$$R_{1,m}^m h = \left(\sum_{n=0}^{N_m-1} \alpha_n Q^n \right)^m h \leq \|h\| \left(\sum_{n=0}^{N_m-1} \alpha_n \right)^m$$

and the sum over m of the right-hand side converges by assumption.

THE BRUNEL EXAMPLE. Let $\|Q^n h\| \rightarrow 0$ and $\sum \|Q^{N_m} h\| < \infty$. Choose

$$\rho_n = (1/n^2)^{1/n}, \quad n \geq 3,$$

then $\rho_n \uparrow 1$ and $\sum \rho_n^n < \infty$. Choose $\alpha_n \geq 0$ such that $\sum_{n=0}^{N_m-1} \alpha_n < \rho_m$ and $\sum \alpha_n = 1$ and, by Theorem 2.3, $\sum \alpha_n Q^n$ is not conservative.

3. Dissipating power series. Let us call a power series $A(z) = \sum_{n=0}^\infty \alpha_n z^n$ dissipating if (1) $\alpha_n \geq 0$, (2) $A(1) = 1$, and (3) there is some conservative operator P with $A(P)$ dissipative. Theorem 1.1 says simply that if $A'(1)$ is finite then A is not dissipating. The main purpose of this section is to establish a converse: namely if $A'(1)$ is infinite then A is dissipating. We first make a slight detour to discuss *renewal sequences*. Recall that $\{u_n\}_{n=1}^\infty$, $0 \leq u_n \leq 1$, is said to be a renewal sequence if there is a sequence $\{f_n\}_{n=1}^\infty$, $f_n \geq 0$, $\sum_{i=1}^\infty f_i \leq 1$ such that

$$(1) \quad u_n = f_n + f_{n-1}u_1 + f_{n-2}u_2 + \dots + f_1 u_{n-1} \quad (n = 1, 2, \dots).$$

Equivalently, if $U(z) = 1 + \sum u_n z^n$, $F(z) = \sum f_n z^n$ then

$$U(z) = F(z)U(z) + 1 \quad \text{or} \quad U(z) = \frac{1}{1 - F(z)}.$$

If $P = (p_{ij})_{i,j=1}^\infty$ is a Markovian transition matrix with all states forming a single ergodic class, the condition for recurrence or conservativeness is simply $\sum_{n=0}^\infty p_{11}^{(n)} = +\infty$ where $p_{ij}^{(n)}$ is the ij entry in P^n . It is well known that $\{p_{11}^{(n)}\}_{n=1}^\infty$ forms a renewal sequence. Here the f_n of (1) represent the probability that first return to 1 takes place at time n . We shall need the simple converse.

LEMMA 3.1. *If $\{u_n\}_{n=1}^\infty$ is a renewal sequence then there is an ergodic Markov matrix with $u_n = p_{11}^{(n)}$.*

PROOF. Let f_n be such that (1) holds and define $p_1 = f_1, \dots, p_n = f_n / (1 - f_1 - f_2 - \dots - f_{n-1})$. Set now

$$\begin{aligned} p_{ij} &= p_i && \text{if } j = 1, \\ &= 1 - p_i && \text{if } j = i + 1, \\ &= 0 && \text{otherwise.} \end{aligned}$$

The lemma now follows easily if one recalls the probabilistic interpretation of the f_j 's, namely that they are the probability that the event occurs for the first time at time n . The structure of p_{ij} is such that one easily checks

$$\text{Prob}\{\text{return to 1 for the first time at time } n|\text{start at 1}\} \\ = (1 - p_1)(1 - p_2) \cdots (1 - p_{n-1})p_n = f_n. \quad \square$$

The existence of a plentiful supply of renewal sequences is assured by Th. Kaluza's theorem [5] to the effect that if $1 \geq u_n \geq 0$ and

$$u_n/u_{n-1} \leq u_{n+1}/u_n, \quad n = 1, 2, \dots \quad (u_0 = 1),$$

then $\{u_n\}$ is a renewal sequence. Indeed as D. G. Kendall [6] has shown, these are precisely the "infinitely divisible" renewal sequences. We shall also need the following lemma, a proof of which may be found in [1].

LEMMA 3.2. *If x_j is a sequence of nonnegative numbers that tend to zero as $j \rightarrow \infty$ then there is a renewal sequence $\{u_n\}$, in fact, an infinitely divisible one, such that $\sum_1^\infty u_n = +\infty$ but $\sum_1^\infty u_n b_n < \infty$.*

THEOREM 3.1. *If $A(z) = \sum_1^\infty \alpha_n z^n$, $\alpha_n \geq 0$, $A(1) = 1$ and $A'(1) = \infty$ then A is dissipating.*

PROOF. Let β_j be defined by

$$\sum_0^\infty \beta_j z^j = \frac{1}{1 - A(z)} = \sum_0^\infty A(z)^n.$$

Then since $A'(1) = 1$ by the renewal theorem (see [2, Chapter XIII.3]) we know that β_j tends to zero. Apply Lemma 3.2 to obtain a renewal sequence with $\sum_1^\infty u_n = \infty$ and $\sum_1^\infty u_j \beta_j < \infty$. By Lemma 3.1 there is a Markov matrix with $p_{11}^{(n)} = u_n$. Thus P is conservative. However, $A(P) = Q$ is dissipative since

$$\sum_0^\infty Q_{11}^{(n)} = \left(\sum_0^\infty Q^n \right)_{11} = \left(\sum_0^\infty A(P)^n \right)_{11} = \left(\sum_0^\infty \beta_n P^n \right)_{11} \\ = \sum_0^\infty \beta_n p_{11}^{(n)} = \sum_0^\infty \beta_n u_n < +\infty.$$

The formal interchanges of summations is justified since all the terms are nonnegative and the final result is a finite quantity. \square

It is worth remarking that even when a conservative operator P has no finite invariant measure there are dissipating power series $A(z)$ such that $A(P)$ is conservative. To see this it suffices to consider the special translation invariant Markov chains on the integers Z —the random walks

defined by $\{p_j\}$, a probability distribution on Z . A necessary and sufficient condition for recurrence is known here in terms of $\varphi(v) = \sum_{-\infty}^{\infty} p_n e^{i n v}$, the characteristic function of φ , namely

$$\int_{-\pi}^{+\pi} \operatorname{Re} \left(\frac{1}{1 - \varphi(v)} \right) dv = +\infty \quad [9, \text{Chapter II.8}].$$

Picking p_j with prescribed behavior at infinity and using a Tauberian theorem to relate the behavior of $\varphi(v)$ at $v=0$ one readily produces examples for the phenomenon described above.

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