PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 38, Number 2, April 1973

THE FOURIER TRANSFORM IS ONTO ONLY WHEN THE GROUP IS FINITE

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ABSTRACT. A very simple proof of the result of the title is given. Unlike previous proofs, the one presented here uses no results of harmonic analysis beyond the Pontryagin duality theorem.

THEOREM. *Let G be a LCA group with dual $\hat{G}$. If the Fourier transform*

$$\mathscr{F}: L^1(G) \to C_0(\hat{G})$$

is onto, then G is finite.

This result was proved in [2], [3] and more recently by Friedberg [1]. We give an even simpler proof than those of [1], [2], [3]. I am grateful to the referee for calling my attention to [4], which contains a proof of the corresponding result for some noncommutative groups.

PROOF. From the Pontryagin duality theorem and the closed graph theorem, we know $\mathscr{F}$ is an isomorphism of Banach spaces, so $\mathscr{F}^*$ will also be an isomorphism of the corresponding dual spaces $\mathscr{F}^*: M(\hat{G}) \to L^\infty(G)$. A simple computation shows that if $\mu \in M(G)$ then $\mathscr{F}^*\mu(x) = \hat{\mu}(x)$ almost everywhere, where $\hat{\mu}$ is the Fourier-Stieltjes transform of μ. But if G is not discrete, and $U \subseteq G$ is an opening, relatively compact non-dense set, then the characteristic function of U is not equal almost everywhere to a continuous function. Since each $\hat{\mu}$ ($\mu \in M(\hat{G})$) is continuous, G can only be discrete and $\mathscr{F}^*\mu = \hat{\mu}$ everywhere.

We consider $\mathscr{F}^*$ restricted to $L^1(\hat{G})$. Because $\mathscr{F}^*$ is an isomorphism of Banach spaces, $\mathscr{F}^*L^1(\hat{G})$ is uniformly closed in $L^\infty(\hat{G})$. Of course, $\mathscr{F}^*L^1(\hat{G})$ is also dense in $C_0(G)$. Thus $\mathscr{F}^*: L^1(\hat{G}) \to C_0(G)$ is onto. By the argument of the first paragraph, $\hat{G}$ is discrete. Thus G is discrete and compact so G is finite. Q.E.D.

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Received by the editors February 28, 1971 and, in revised form, July 19, 1972.

AMS (MOS) subject classifications (1970). Primary 42A68, 43A25; Secondary 46J10.

[1] Partially supported by National Science Foundation Grant GP-32116.

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