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# A CHARACTERIZATION OF THE BROWDER SPECTRUM

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ABSTRACT. Several different notions of essential spectra are known. This paper is devoted to giving an equivalent definition of the *Browder* essential spectrum for bounded operators on Hilbert space. Also, it is shown that the set of all operators with essentially disconnected spectrum is a uniformly dense set.

1. **Introduction.** Throughout this paper $\mathscr{H}$ will denote a fixed infinite dimensional, complex, Hilbert space and $\mathscr{L}(\mathscr{H})$ will represent the algebra of all (bounded, linear) operators on $\mathscr{H}$. If $T\in\mathscr{L}(\mathscr{H})$, then the Browder spectrum $B(T)$ of $T$ can be defined [**7**, §2] as the complement of those complex numbers $\lambda$ such that $T-\lambda$ is a Fredholm operator of index zero and $\lambda$ is an isolated point of the spectrum $\Sigma(T)$ of $T$. With the terminology of [**4**, Chapter 1, §2], $B(T)$ is the complement of the set of normal points of $T$. This concept, due to F. Browder [**2**, §6], has recently been discussed in several papers [**6**], [**7**], [**8**] and turned out to be an important characteristic of an operator.

Following [**7**], we shall denote by $(ED)$ the set of all operators $T\in\mathscr{L}(\mathscr{H})$ such that the polynomial hull $\hat{B}(T)$ of $B(T)$ is disconnected. (We recall that the polynomial hull $\hat{X}$ of a compact subset $X$ of the complex plane $\boldsymbol{C}$ is the complement of the unbounded component of $\boldsymbol{C}-X$.) In [**7**], it is shown that $(ED)$ is a uniformly open subset of $\mathscr{L}(\mathscr{H})$ invariant under compact perturbations. Moreover, if $T\in(ED)$, then $\hat{\Sigma}(T+K)$ is disconnected, for every compact operator $K$. For this reason operators in $(ED)$ are called operators with essentially disconnected spectrum. On the other hand, minor modifications of the arguments given in [**9**] show that if $T$ is any operator in $\mathscr{L}(\mathscr{H})$, then there exists a compact operator $K$ on $\mathscr{H}$ such that $\Sigma(T+K)=B(T)$. From this result it follows that $T\in(ED)$ *if and only if* $\hat{\Sigma}(T+K)$ is disconnected for every compact operator $K$.

In the present paper, we give a characterization of the Browder spectrum which is independent of Fredholm theory (Theorem 2.1), answering a

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question raised in [**7**, §5]. Also, we prove that the set $(ED)$ is dense in $\mathscr{L}(\mathscr{H})$ in the uniform topology.

2. **Compression to subspaces of finite codimension.** We begin our discussion introducing some terminology. If $T$ is any operator on $\mathscr{H}$ and $Q$ is any (orthogonal) projection in $\mathscr{L}(\mathscr{H})$, we shall denote by $T_Q$ the compression of $T$ to the range of $Q$, i.e. $T_Q=(QT)|_{Q\mathscr{H}}$. Also, $\mathscr{P}$ will represent the set of all projections in $\mathscr{L}(\mathscr{H})$ whose null space is finite dimensional.

The main result of this section can be stated as follows.

THEOREM 2.1. *If $T\in\mathscr{L}(\mathscr{H})$, then*

$$B(T)=\bigcap_{Q\in\mathscr{P}}\Sigma(T_Q).$$

The proof of this theorem requires some auxiliary results and will be given later.

The following lemma is central to our purposes.

LEMMA 2.2. *Let $T\in\mathscr{L}(\mathscr{H})$ and $Q\in\mathscr{P}$. Then $\Sigma(T)-\Sigma(T_Q)$ is a subset of $\boldsymbol{C}$ consisting only of isolated points.*

PROOF. Let $\mathscr{N}=Q\mathscr{H}$, and let $\mathscr{M}=\mathscr{H}\ominus\mathscr{N}$. Then $T$ can be represented by a $2\times 2$ operator matrix of the form

$$T=\begin{bmatrix}A & B\\ C & D\end{bmatrix}$$

acting in the usual matricial fashion on $\mathscr{M}\oplus\mathscr{N}$ (note that $D=T_Q$). If $\lambda\in\boldsymbol{C}-\Sigma(T_Q)$, then it follows from [**3**, Lemma 3.2] that $T-\lambda$ is invertible if and only if the same property holds for the operator $S(\lambda)=(A-\lambda)-B(D-\lambda)^{-1}C$, acting on the finite dimensional space $\mathscr{M}$. For each $\lambda\in\boldsymbol{C}-\Sigma(T_Q)$, let $d(\lambda)$ be the determinant of the representing matrix of $S(\lambda)$, with respect to some fixed orthonormal basis of $\mathscr{M}$. Since $S(\lambda)$ is an analytic operator valued function for $\lambda\in\boldsymbol{C}-\Sigma(T_Q)$, it follows that $d$ is an analytic complex valued function on the same region and from the preceding remark we conclude that $\Sigma(T)-\Sigma(T_Q)$ coincides with the set of zeros of the function $d$. Now the lemma follows from a well-known property of analytic functions.

REMARK 2.3. Another important concept native to the theory of compact perturbations is the Weyl spectrum $\Omega(T)$ of the operator $T$ [**1**]. $\Omega(T)$ can be defined as the complement of those complex numbers $\lambda$ such that $T-\lambda$ is a Fredholm operator of index zero. Since an operator is Fredholm of index zero if and only if so is any of its compact perturbations, it follows that $\Omega(T)=\Omega(T+K)$, for every $T\in\mathscr{L}(\mathscr{H})$ and every compact operator $K$ on $\mathscr{H}$. In particular, if $Q\in\mathscr{P}$, then $T-QTQ$ is a compact operator and hence $\Omega(T)=\Omega(QTQ)=\Omega(T_Q)$. It is clear from the definition of $B(T)$ and $\Omega(T)$ that $\Omega(T)\subseteq B(T)$. It can be shown [7, Theorem 1] that $B(T)-\Omega(T)$ can only consist of some of the holes (bounded components of the complement) of $\Omega(T)$. Thus, we deduce that for every $Q\in\mathscr{P}$ the sets $B(T)$ and $B(T_Q)$ can only differ in some of the holes of $\Omega(T)$. The following lemma shows that a spectral inclusion formula actually holds.

LEMMA 2.4. *If $Q\in\mathscr{P}$ and $T$ is an operator on $\mathscr{H}$, then $B(T)\subseteq B(T_Q)$.*

PROOF. Suppose (on the contrary) $B(T)-B(T_Q)\neq\varnothing$. From the above remark it follows that $B(T)-B(T_Q)$ is a set of nonisolated points. Since $\Sigma(T_Q)-B(T_Q)$ consists only of isolated points and $(B(T)-B(T_Q))-(\Sigma(T_Q)-B(T_Q))=B(T)-\Sigma(T_Q)\subseteq\Sigma(T)-\Sigma(T_Q)$ we conclude that $\Sigma(T)-\Sigma(T_Q)$ contains some nonisolated points. This contradicts Lemma 2.2.

REMARK 2.5. The following example shows that the inclusion in Lemma 2.4 may be a proper inclusion. Suppose that $\mathscr{H}$ is a separable Hilbert space and let $\{e_n\}$, $n=0, \pm 1, \pm 2, \cdots$, be an orthonormal basis of $\mathscr{H}$. Let $U$ be the simple bilaterial shift on $\mathscr{H}$ defined by the relation $Ue_n=e_{n+1}$, $n=0, \pm 1, \pm 2, \cdots$. Also let $Q$ be the projection onto the orthogonal complement of the subspace generated by $e_0$. We see that $U_Qe_n=e_{n+1}$, $n\geqq 1$, $U_Qe_{-n}=e_{-n+1}$, $n\geqq 2$, $U_Qe_{-1}=0$. Therefore $U_Q$ is unitarily equivalent to $V\oplus V^*$, where $V$ is a simple unilateral shift on $\mathscr{H}$. It follows easily that $B(V\oplus V^*)$ is the closed unit disc while $B(U)$ is its boundary. Thus, we conclude $B(U)$ is properly contained in $B(U_Q)$, as desired.

PROOF OF THEOREM 2.1. From Lemma 2.4 we know that $B(T)\subseteq\Sigma(T_Q)$, for all $Q\in\mathscr{P}$. Hence $B(T)\subseteq\bigcap_{Q\in\mathscr{P}}\Sigma(T_Q)$. To prove the other inclusion let $\mathscr{G}$ be any open neighborhood of $B(T)$. Since $B(T)$ contains all the accumulation points of $\Sigma(T)$, it follows that $\Sigma(T)-\mathscr{G}$ is a finite set of points $\lambda$ such that $T-\lambda$ is Fredholm of index zero. Let $E$ be the spectral idempotent associated with the closed and open subset $\Sigma(T)\cap\mathscr{G}$ of $\Sigma(T)$. It is well known that range $E$ is an invariant subspace of $T$ and $\Sigma(T|_{(\text{range } E)})=\Sigma(T)\cap\mathscr{G}$. Let $Q$ be the projection onto range $E$. Then $\Sigma(T_Q)\subseteq\mathscr{G}$ and $Q\in\mathscr{P}$. (Observe that the null space of $E$ is finite dimensional.) Recalling that $\mathscr{G}$ is an arbitrary open neighborhood of $B(T)$ we conclude that $\bigcap_{Q\in\mathscr{P}}\Sigma(T_Q)\subseteq B(T)$. The proof of the theorem is complete.

In [**8**] Schechter observed that the Weyl spectrum of an operator $T$ coincides with the intersection of the spectra of all compact perturbations of $T$ (see also [**1**, §3]). Thus, Theorem 2.1 provides a characterization of the Browder spectrum similar to the characterization of the Weyl spectrum given by Schechter.

3. **A density theorem.** In [**5**], it is proved that the set of all operators with disconnected spectrum is uniformly dense in $\mathscr{L}(\mathscr{H})$. The following theorem represents a substantial improvement of that result.

THEOREM 3.1. *The set (ED) is uniformly dense in* $\mathscr{L}(\mathscr{H})$.

PROOF. Let $T \in \mathscr{L}(\mathscr{H})$ and let $\lambda_0$ be any point in $\Omega(T)$ satisfying Re $\lambda_0 = \max_{\lambda \in \Omega(T)}$ Re $\lambda$. Since $\lambda_0$ lies in the topological boundary of $\Omega(T)$, it follows that either the null space of $T-\lambda_0$ is infinite dimensional, or the range of $T-\lambda_0$ is not closed. From a result of Wolf [**10**] we deduce that there exists an infinite rank projection $P \in \mathscr{L}(\mathscr{H})$ whose null space is infinite dimensional, and such that $(T-\lambda_0)P$ is a compact operator. Let $Q$ be an infinite rank subprojection of $P$ such that $P-Q$ is also an infinite rank projection, and let $\mathscr{K}$, $\mathscr{M}$ and $\mathscr{N}$ be the infinite dimensional subspaces of $\mathscr{H}$ defined by $\mathscr{K}=Q\mathscr{H}$, $\mathscr{M}=(P-Q)\mathscr{H}$, and $\mathscr{N}=\mathscr{H}\ominus(\mathscr{K}\oplus\mathscr{M})$. Then the operator $T$ can be represented by a $3\times 3$ matrix (acting on $\mathscr{K}\oplus\mathscr{M}\oplus\mathscr{N}$) of the form

$$T = \begin{bmatrix} \lambda_0 1_{\mathscr{K}} + K_{11} & K_{12} & * \\ K_{21} & \lambda_0 1_{\mathscr{M}} + K_{22} & * \\ K_{31} & K_{32} & S \end{bmatrix},$$

where $K_{ij}$ is a compact transformation, $i=1, 2, 3, j=1, 2$ and $S$ is an operator on $\mathscr{N}$. It readily follows that $\Omega(T)=\Omega(\lambda_0 1_{\mathscr{K}}\oplus\lambda_0 1_{\mathscr{M}}\oplus S)=\Omega(\lambda_0 1_{\mathscr{M}}\oplus S)$. Let $\varepsilon$ be any positive number. Then

$$\Omega(T + \varepsilon Q) = \Omega([\lambda_0 + \varepsilon]1_{\mathscr{K}} \oplus \lambda_0 1_{\mathscr{M}} \oplus S) = \{\lambda_0 + \varepsilon\} \cup \Omega(T).$$

Since $\hat{B}(T+\varepsilon Q)=\hat{\Omega}(T+\varepsilon Q)=\{\lambda_0+\varepsilon\}\cup\hat{\Omega}(T)=\{\lambda_0+\varepsilon\}\cup\hat{B}(T)$ and $\lambda_0+\varepsilon\notin \hat{B}(T)$, we conclude that $\hat{B}(T+\varepsilon Q)$ is disconnected and the proof of the theorem is complete.

We say that a subspace $\mathscr{M}$ of $\mathscr{H}$ is hyperinvariant for an operator $T$ on $\mathscr{H}$ if $\mathscr{M}$ is invariant under every operator in $\mathscr{L}(\mathscr{H})$ commuting with $T$. Employing arguments similar to those used in the proof of [**5**, Corollary 2], the following consequence of Theorem 3.1 can be obtained.

COROLLARY 3.2. *Let n be any cardinal number such that* $1\leqq n\leqq \aleph_0$, *and* let $\mathscr{S}_n$ *be the set of operators in* $\mathscr{L}(\mathscr{H})$ *having two complementary hyperinvariant subspaces* $\mathscr{M}$ *and* $\mathscr{N}$ *such that* dim $\mathscr{M}=n$ *and* dim $\mathscr{N}=$dim $\mathscr{H}$. *Then* $\mathscr{S}_n$ *contains a uniformly open dense subset of* $\mathscr{L}(\mathscr{H})$. *Thus,* $\bigcap_{1\leqq n\leqq\aleph_0}\mathscr{S}_n$ *is uniformly dense in* $\mathscr{L}(\mathscr{H})$.

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