PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 38, Number 2, April 1973

PERFECT MAPS OF SYMMETRIZABLE SPACES

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ABSTRACT. It is shown that if $f:X\to Y$ is a perfect map from a symmetrizable space X onto a space Y, then Y is metrizable if and only if f is a coherent map. This fact, together with certain known results, yields the following: Let $f:X\to Y$ be a perfect map from a Hausdorff symmetrizable space X onto a space Y; the following are equivalent: (1) X is metrizable; (2) f is a regular map; (3) f is a coherent map; (4) Y is metrizable.

A topological space X is said to be *symmetrizable* if there exists a nonnegative real valued function d on $X\times X$, called a *symmetric*, which satisfies the following three conditions: (1) $d(a, b)=0$ if and only if $a=b$; (2) $d(a, b)=d(b, a)$; (3) a subset A of X is closed if and only if whenever $x\in X-A$, then $d(x, A)>0$.

A function $f:X\to Y$ from a space X onto a space Y is said to be *coherent* if the space X is symmetrizable via a symmetric d such that whenever $\{a_n\}$ and $\{b_n\}$ are sequences in X with $d(a_n, b_n)\to 0$ and $f(a_n)\to y$ in Y, then $f(b_n)\to y$. Coherent maps are closely related to the regular maps of A. Arhangel'skiĭ [**1**, p. 133]. Every regular map is a coherent map. The extent of coherent maps may be seen in the following, which is not difficult to prove: Let $f:X\to Y$ be a function from a symmetrizable space X onto a metrizable space Y; then, f is continuous if and only if f is a coherent map.

A map is *perfect* if it is closed, continuous and point inverses are compact, i.e., bicompact, sets.

THEOREM 1. *Let $f:X\to Y$ be a perfect map from a symmetrizable space X onto a space Y. Then, Y is metrizable if and only if f is a coherent map.*

PROOF. Assume that Y is metrizable. Let ρ be a symmetric for X and d be a metric for Y. For points a and b in X, let $\sigma(a, b)=\rho(a, b)+d(f(a), f(b))$. σ is a symmetric compatible with the topology for X. It is easy to verify that f is a coherent map by virtue of the symmetric σ.

Presented to the Society, January 28, 1972 under the title *Perfect coherent maps of symmetrizable spaces*; received by the editors April 10, 1972 and, in revised form, July 31, 1972.

AMS (MOS) subject classifications (1970). Primary 54C10, 54E35; Secondary 54E25.

Key words and phrases. Metrizable space, symmetrizable space, symmetric, G_δ-diagonal, perfect map, coherent map, regular map.

To prove the converse, assume that f is a coherent map by virtue of a symmetric ρ for the space X. For $a, b \in Y$, define $d(a, b)=\rho(f^{-1}(a), f^{-1}(b))$. We shall show that the space Y is symmetrizable via d. Clearly, $d(a, b)=d(b, a)$ for all $a, b \in Y$. That $d(a, b)=0$ if and only if $a=b$ follows easily from the fact that f is a coherent map and Y is a T_1 space. Let A be a closed subset of Y and $y \in Y-A$. If $d(y, A)=0$, then $\rho(f^{-1}(y), f^{-1}[A])=0$ and there would exist sequences $\{b_n\}$ in $f^{-1}(y)$ and $\{a_n\}$ in $f^{-1}[A]$ such that $\rho(a_n, b_n)\to 0$; since $f(b_n)\to y$, we would have $f(a_n)\to y$, contradicting the closedness of A. Consequently, $d(y, A)>0$. Finally, assume that B is not a closed subset of Y. Since f is a quotient map, $f^{-1}[B]$ is not a closed subset of X. Therefore, there exists $x \in X-f^{-1}[B]$ such that $\rho(x, f^{-1}[B])=0$. Then $d(f(x), B)=0$, and we have completed the proof that the space Y is symmetrizable via d.

Let A be a compact subset of Y and B be a closed subset of Y such that $d(A, B)=0$. Then $\rho(f^{-1}[A], f^{-1}[B])=0$ and since $f^{-1}[A]$ is compact, there exists $x \in f^{-1}[A]$ and sequences $\{a_n\}$ in $f^{-1}[A]$ and $\{b_n\}$ in $f^{-1}[B]$ such that $a_n\to x$ and $\rho(a_n, b_n)\to 0$. Since f is a coherent map, we have $f(b_n)\to f(x)$. Since B is closed, it follows that $f(x) \in B$ so that A and B are not disjoint. Hence, if A and B are disjoint subsets of Y with A compact and B closed, then $d(A, B)>0$. By Theorem 2 of [**3**], d is a coherent distance function. The metrizability of Y now follows by a theorem of Niemytzki and Wilson [**5**], [**8**], completing the proof.

If a Hausdorff space X maps perfectly onto a metrizable space, then X is metrizable if and only if X has a G_δ-diagonal [**2**], [**6**]. Recall also that the perfect image of a metrizable space is metrizable [**4**], [**7**]. These facts, together with Theorem 1, yield the following:

THEOREM 2. *Let $f:X\to Y$ be a perfect map from a symmetrizable Hausdorff space X onto a space Y. The following are equivalent*:

(1) *X is metrizable.*

(2) *f is a regular map.*

(3) *f is a coherent map.*

(4) *Y is metrizable.*

PROOF. Assume that X is metrizable. Then Y is metrizable. It follows that f is regular [**1**, p. 134], so that (1) implies (2). As seen in [**3**], every regular map is coherent, so that (2) implies (3). (3) implies (4) by Theorem 1. Finally, assume that Y is metrizable. Then X is regular; any regular space which maps onto a first countable space by a closed map with first countable point inverses is itself first countable, i.e., X is a first countable space. This completes the proof since any first countable symmetrizable Hausdorff space has a G_δ-diagonal.

The Hausdorff condition on X in Theorem 2 cannot be completely removed since there exist non-Hausdorff compact symmetrizable spaces.

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