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AN INTEGRAL EQUATION ARISING IN POTENTIAL THEORY[1]

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ABSTRACT. This paper gives an integral equation, the solution of which is a solution of a classical problem in potential theory: Given a region with boundary $\mathscr{B}$, what distribution of charge on $\mathscr{B}$ will produce a potential having specified values on $\mathscr{B}$? The paper also indicates briefly how the integral equation is useful in simplifying certain proofs and extending certain theorems in potential theory.

The definitions and notation of this paper are those of [2, pp. 353–355].

Let α be a function having a derivative $D\alpha$ which satisfies a Hölder condition on $\mathscr{B}$. Let Ψ be the potential due to the double-layer distribution whose moment-density is α; that is, let

$$\Psi z = \int_0^{s_m} \alpha s \cdot D_{u_s}\Gamma(\xi s - x, \eta s - y)\, ds \qquad (z = x + iy),$$

for each z in the plane, except those on $\mathscr{B}$. Then [1, pp. 42–46] the normal derivative of Ψ is continuous on $\mathscr{B}_i \cup \mathscr{B}$, where $\mathscr{B}_i$ denotes the union of the interior regions determined by $\mathscr{B}$. An expression for the limit of this normal derivative at the point ζs on $\mathscr{B}$ is the Cauchy principal value integral $\int_{\mathscr{I}} (D\alpha \cdot D_1\Lambda(s, \iota))$, where $D_1\Lambda$ is the first-place partial derivative of Λ. The function whose value at s is this integral will be denoted by $S\alpha$.

LEMMA 1. *There exists a continuous function* $\bar{\nu}_1$ *such that* $\lim_{z\to\zeta s} D_{u_s}\bar{\Phi}_i z = S\alpha s$, *where* $\bar{\Phi}_i z = \int_{\mathscr{I}} (\bar{\nu}_1 \cdot \log(c/|\zeta - z|))$ *for all* z *in* $\mathscr{B}_i$.

PROOF. The function $\bar{\nu}_1$ with the property stated exists if the equation $\bar{\nu}_1 - T\bar{\nu}_1 = -S\alpha/\pi$ has a continuous solution. Since the operator T is compact in $\mathscr{L}^2[0, s_m]$, it follows by the Fredholm theory that this equation has a solution if and only if $\int_{\mathscr{I}} (S\alpha \cdot \psi_j) = 0$ for $j = 1, \cdots, m$, where $\psi_1, \cdots, \psi_m$

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are m linearly independent solutions of the equation $\psi_j - \bar{T}\psi_j = 0$, the operator $\bar{T}$ being defined for each α in $\mathscr{P}$ by the equality $\bar{T}\alpha s = \int_{\mathscr{I}} (\alpha \cdot K(s, \iota))$. The functions $\psi_1, \cdots, \psi_m$ [2, equation (2)] defined by the relations

$$\begin{aligned} \psi_j s &= 1/c_j \quad \text{if } \zeta s \in \mathscr{B}_j, \\ &= 0 \qquad \text{if } \zeta s \in \mathscr{B}_k,\ k \neq j, \end{aligned}$$

are such solutions, so the conditions $\int_{\mathscr{I}} (S\alpha \cdot \psi_j) = 0$ are equivalent to the conditions $\int_{s_{j-1}}^{s_j} S\alpha s\, ds = 0$, which, by Green's Second Identity, are satisfied because $S\alpha$ is the normal derivative on $\mathscr{B}$ of a function harmonic in $\mathscr{B}_i$. Then every function of the form $\bar{\nu}_1 + \sum_{j=1}^m a_j \varphi_j$, where $a_1, \cdots, a_m$ are numbers, is a solution of the equation $\gamma - T\gamma = -S\alpha/\pi$. Hence every function whose value at each point z of $\mathscr{B}_i$ is $\int_{\mathscr{I}} ((\bar{\nu}_1 + \sum_{j=1}^m a_j\varphi_j) \cdot \log(c/|\zeta - z|))$ has the property that the limit of its normal derivative at ζs is $S\alpha s$. That $\bar{\nu}_1$ is continuous on $\mathscr{B}$ follows from the form of the solution given by the Fredholm theory, when one takes into account property (iv), p. 362, of [2] and the fact that $S\alpha$ is continuous. Since every φ_j satisfies a Hölder condition on $\mathscr{B}$, every solution given above is continuous on $\mathscr{B}$.

LEMMA 2. *There exists a continuous function* ν_1 *such that* $\Psi z = \int_{\mathscr{I}} (\nu_1 \cdot \log(c/|\zeta - z|))$ *for all* z *in* $\mathscr{B}_i$.

PROOF. Let $\bar{\nu}_1$ be a particular function whose existence is assured by Lemma 1, and let $\bar{\Phi}_i$ be defined as in that lemma. Then $\lim_{z\to\zeta s} D_{u_s}\Psi z = \lim_{z\to\zeta s} D_{u_s}\bar{\Phi}_i z$, and hence, by the uniqueness, to within an additive constant, of the solution of the interior Neumann problem, there exists a function Δ_i such that $\Psi = \bar{\Phi}_i + \Delta_i$ and, for some numbers $d_1, \cdots, d_m$, $\Delta_i z = d_k$ if z is in the interior of $\mathscr{B}_k$, for $k = 1, \cdots, m$. If $a_1, \cdots, a_m$ are chosen so that

$$a_j \int_{\mathscr{I}} (\varphi_j \cdot \log(c/|\zeta - z|)) = -d_j,$$

then

$$\Psi z = \int_{\mathscr{I}} (\nu_1 \cdot \log(c/|\zeta - z|))$$

for all z in $\mathscr{B}_i$, where $\nu_1 = \bar{\nu}_1 + \sum_{j=1}^m a_j\varphi_j$.

LEMMA 3. *There exists a continuous function* ν_2 *such that* $\Phi_e z = \Psi z$ *for all* z *in the exterior*, $\mathscr{B}_e$, *of* $\mathscr{B}$, *where* $\Phi_e z = \int_{\mathscr{I}} (\nu_2 \cdot \log(c/|\zeta - z|))$.

PROOF. There exists a function ν_2 such that $\lim_{z\to\zeta s} D_{u_s}\Phi_e z = S\alpha s$ if the equation $\nu_2 + T\nu_2 = S\alpha/\pi$ has a solution. It does, in fact, have a unique solution ν_2, as can be seen by noting that the equation $\gamma + \bar{T}\gamma = 0$ has no nontrivial solutions and then applying the Fredholm theory in $\mathscr{L}^2[0, s_m]$.

Hence, by the uniqueness, to within an additive constant, of the solution of the exterior Neumann problem, there is a constant function Δ_e such that $\Psi z = \Phi_e z + \Delta_e z$ for all z in $\mathscr{B}_e$.

Since Ψ is a double-layer potential, $\lim_\infty \Psi = 0$. Since the masses of Ψ and Δ_e are both zero, the mass of Φ_e is zero, which implies that $\lim_\infty \Phi_e = 0$ because Φ_e is a single-layer potential. Therefore $\lim_\infty \Delta_e = 0$, so that $\Delta_e = 0$, and hence $\Psi = \Phi_e$ on $\mathscr{B}_e$.

The continuity of ν_2 can be established by a proof similar to that of the continuity of ν_1 in Lemma 1.

LEMMA 4. *If α, ν_1, and ν_2 are defined as in the preceding lemmas, then*

$$(1) \qquad 2\pi\bar{T}\alpha = L(\nu_1 + \nu_2)$$

and

$$(2) \qquad 2\pi\alpha = L(\nu_1 - \nu_2),$$

where L is the operator defined for each γ in $\mathscr{P}$ by $L\gamma t = \int_{\mathscr{I}} (\gamma \cdot \Lambda(\iota, t))$.

PROOF. If Ψ_i and Ψ_e denote the restrictions of Ψ to $\mathscr{B}_i$ and $\mathscr{B}_e$, respectively, then by Lemmas 2 and 3, and by the well-known boundary behavior of a double-layer potential,

$$L\nu_1 s = \lim_{z \to \zeta s} \Psi_i z = \pi\alpha s + \pi\bar{T}\alpha s$$

and

$$L\nu_2 s = \lim_{z \to \zeta s} \Psi_e z = -\pi\alpha s + \pi\bar{T}\alpha s.$$

Adding these equations gives (1) and subtracting them gives (2).

LEMMA 5. *If α, ν_1, and ν_2 are defined as in the preceding lemmas, then*

$$(3) \qquad \nu_1 - \nu_2 - T(\nu_1 + \nu_2) = -(2/\pi)S\alpha$$

and

$$(4) \qquad T(\nu_1 - \nu_2) = \nu_1 + \nu_2.$$

PROOF. On the one hand, using the representations of Ψ_e and Ψ_i as single-layer potentials given in Lemmas 1 and 2, $\lim_{z \to \zeta s} D_{u_s} \Psi_i z = -\pi\nu_1 s + \pi T\nu_1 s$ and $\lim_{z \to \zeta s} D_{u_s} \Psi_e z = \pi\nu_2 s + \pi T\nu_2 s$; on the other hand, using the definition of Ψ as a double-layer potential, $\lim_{z \to \zeta s} D_{u_s} \Psi_i z = \lim_{z \to \zeta s} D_{u_s} \Psi_e z = S\alpha s$. Adding the first two of these equalities and taking account of the third gives (3); subtracting the first two and taking account of the third gives (4).

THEOREM 1. *If α is a function such that $D\alpha$ satisfies a Hölder condition on $\mathscr{B}$, then there is a continuous function β such that $\alpha = L\beta$ and*

$$(T^2 - I)\beta = (1/\pi^2)S\alpha. \tag{5}$$

PROOF. The existence of the function β is given by Lemma 4 by taking β to be $(\nu_1 - \nu_2)/2\pi$. Operating on equation (4) with T and using the result in equation (3) gives equation (5).

REMARK. Operating on equation (5) with L and making use of the fact that $LT^2 = \bar{T}^2 L$ gives the equality $LS\alpha = \pi^2(\bar{T}^2 - I)\alpha$, valid for all α satisfying the hypotheses of the theorem.

The following lemma, an extension of a result obtained by Kellogg [**1**, p. 46, footnote], gives an alternative characterization of the operator S.

LEMMA 6. *If α is a function such that $D\alpha$ satisfies a Hölder condition on $\mathscr{B}$, then $S\alpha = DLD\alpha$.*

PROOF. For each s in $\mathscr{I}$ and each sufficiently small positive number e, let $\mathscr{E} = \{t : |A(s, t)| < e\}$. Since $D\alpha$ and $D_1\Lambda(s, \iota)$ are continuous on $\mathscr{I} - \mathscr{E}$, it follows that

$$DLD\alpha s = \lim_{e \to 0^+} \int_{\mathscr{I}-\mathscr{E}} (D\alpha \cdot D_1\Lambda(s, \iota)),$$

provided that the limit exists and the convergence to the limit is uniform with respect to s.

To establish these facts, note that

$$\begin{aligned} \int_{\mathscr{I}-\mathscr{E}} (D\alpha \cdot D_1\Lambda(s, \iota)) &= \int_{\mathscr{I}-\mathscr{E}} (D\alpha t - D\alpha s)(D_1\Lambda(s, t))\, dt \\ &\quad + (D\alpha s) \int_{\mathscr{I}-\mathscr{E}} D_1\Lambda(s, t)\, dt. \end{aligned} \tag{6}$$

Since $D\alpha$ satisfies a Hölder condition on $\mathscr{B}$ with exponent d, say, and since $D_1\Lambda$ is continuous on $\mathscr{I} \times \mathscr{I}$ except at points (s, t) where $A(s, t) = 0$, where it behaves like $|A|^{-1}$, it follows that the integrand of the first integral above behaves like $|A(s, t)|^{d-1}$, and hence the corresponding integral over $\mathscr{I}$ exists in the sense of Lebesgue. Moreover, the convergence of the first integral is uniform, being like that of $|A(s+e, s)|^d$. Now

$$D_1\Lambda(s, t) = -\frac{X(s, t)D_1X(s, t) + Y(s, t)D_1Y(s, t)}{X^2(s, t) + Y^2(s, t)} + \frac{1}{A(s, t)},$$

where X and Y, defined on p. 363 of [**2**], have the properties that $X(s, \iota)$ and $Y(s, \iota)$ satisfy a Hölder condition on $\mathscr{B}$ with the same exponent b as of that satisfied by $D\zeta$ and that $D_1X(s, \iota)$ and $D_1Y(s, \iota)$ are continuous except at points t where $A(s, t)=0$, where they behave like $|A(s, t)|^{b-1}$. Therefore, for some $a>0$,

$$\left|\int_{\mathscr{S}-\mathscr{E}} D_1\Lambda(s, t)\, dt\right| < a(|A(s, s-e)|^b + |A(s, s+e)|^b) + |\log|A(s, s-e)| - \log|A(s, s+e)||,$$

from which both the existence of the limit and the uniform convergence for the second integral in (6) follow. This completes the proof of the lemma.

Theorem 1 and Lemma 6 can be used to give a more elegant proof of the existence and properties of the function Ω_n than was given in [2]. Finally, two other applications of Theorem 1 will be given in the next two theorems.

LEMMA 7. *If α and β are functions as in Theorem 1, then $\langle\beta, S\alpha\rangle = -\|D\alpha\|^2$.*

PROOF. Since the function $DLD\alpha = S\alpha$ is continuous on $\mathscr{B}$,

$$\langle\beta, S\alpha\rangle = \int_{\mathscr{S}} (\alpha \cdot DLD\alpha) = -\int_{\mathscr{S}} (D\alpha \cdot LD\alpha) = -\|D\alpha\|^2.$$

THEOREM 2. *Let $\mathscr{H}_0$ be the subspace of $\mathscr{H}$ which is orthogonal to the functions $\varphi_1, \cdots, \varphi_m$, let $\|T^2\|_0$ be the norm of T^2 on $\mathscr{H}_0$, let α be a function such that $D\alpha$ satisfies a Hölder condition on $\mathscr{B}$, let β be the function such that $\alpha = L\beta$, whose existence is assured by Lemma 4, let β_0 be the projection of β on $\mathscr{H}_0$, and let $\beta_1 = \beta - \beta_0$. Then*

$$\int_{\mathscr{S}} \alpha^2 \leqq \|\alpha\| \left(\|\beta_1\|^2 + \frac{\|D\alpha\|^2}{\pi^2(1 - \|T^2\|_0)}\right)^{1/2}.$$

PROOF. From Theorem 1 and Lemma 7, $\langle T^2\beta, \beta\rangle - \|\beta\|^2 = -\|D\alpha\|^2/\pi^2$. Taking account of the facts that $\|\beta\|^2 = \|\beta_1\|^2 + \|\beta_0\|^2$, that $\langle T^2\beta_0, \beta_1\rangle = \langle\beta_0, T^2\beta_1\rangle = 0$, and that $T^2\beta_1 = T^2(\sum_{j=1}^m \langle\beta, \varphi_j\rangle\varphi_j) = \sum_{j=1}^m \langle\beta, \varphi_j\rangle\varphi_j = \beta_1$, the above equality yields $\langle T^2\beta_0, \beta_0\rangle - \|\beta_0\|^2 = -\|D\alpha\|^2/\pi^2$, or $\|\beta_0\|^2 \leqq \|D\alpha\|^2/\pi^2(1 - \|T^2\|_0)$. Finally,

$$\int_{\mathscr{S}} \alpha^2 = \langle\alpha, \beta\rangle \leqq \|\alpha\| \cdot \|\beta\| \leqq \|\alpha\| \, (\|\beta_1\|^2 + \|D\alpha\|^2/\pi^2(1 - \|T^2\|_0))^{1/2}.$$

The following result was obtained by Warschawski [3, p. 11] by using complex function theory. The proof given here would therefore be useful in extending his results to higher dimensions.

THEOREM 3. *For $j = m+1, m+2, \cdots$, let φ_j be the eigenfunction of T associated with the eigenvalue λ_j, and let $\psi_j = L\varphi_j$. Then*

$$\|D\psi_j\|^2 = (\pi^2(\lambda_j^2 - 1)/\lambda_j^2) \, \|\varphi_j\|^2.$$

PROOF. From equation (5) and Lemma 7 the result is obtained by noting that $\langle T^2\varphi_j, \varphi_j\rangle = \|\varphi_j\|^2/\lambda_j^2$.

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