PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 38, Number 2, April 1973

ODD DIMENSIONAL MANIFOLDS WITH REGULAR CONJUGATE LOCUS

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ABSTRACT. We show that all odd dimensional manifolds, for which the first conjugate locus, with respect to some point, is regular, are homeomorphic to a sphere.

In [2] Warner classifies, almost completely, simply connected, complete Riemannian manifolds M^n for which there exists a point $p \in M^n$ whose first conjugate locus is regular. *Almost completely* means that he needs one of the following two conditions:

(1) The first conjugate point with respect to p in any direction has order $k \geqq 2$.

(2) All the first conjugate points with respect to p lie at the same distance from p.

The purpose of this note is to observe that in the odd dimensional case neither assumption is necessary.

PROPOSITION. *Let M^n be a complete simply connected Riemannian manifold, n odd, and suppose there exists $p \in M^n$ such that the first conjugate point with respect to p exists in any direction and has constant order k. Then $k=n-1$ and M^n is homeomorphic to S^n.*

PROOF. If we show that $k>1$ then Warner's condition (1) is satisfied and n odd implies $k=n-1$ and M^n homeomorphic to S^n. (See [2].)

Suppose $k=1$. Let $C(p)$ denote the first conjugate locus with respect to p. Then $C(p)$ is a smooth closed submanifold of M_p diffeomorphic to an even $(n-1)$-dimensional sphere and transverse to the lines through the origin in M_p (see [1]). For $x \in C(p)$, $\mathrm{Ker}(d\exp_p)_x$ is orthogonal to the line $\{tx \mid t \in R\}$ by Gauss' lemma and therefore has a nontrivial projection on the tangent space $C(p)_x$. In this way we can define a 1-dimensional distribution on $C(p)$, i.e., a 1-dimensional tangent line bundle that is trivial, since $C(p)$ is diffeomorphic to a sphere, and therefore define a nowhere zero vector field. But this is impossible since $n-1$ is even.

Received by the editors August 17, 1972.
AMS (MOS) subject classifications (1970). Primary 53C20.
Key words and phrases. Riemannian manifold, regular conjugate locus.

REMARK. It is easily seen that a manifold for which there exists a point whose first conjugate locus has order $n-1$ is diffeomorphic to the union of two disks. Conversely, if $M^n = D^n \cup_g D^n$, $g: \partial D^n \to \partial D^n$ being a diffeomorphism, it is always possible to put a metric on M^n such that there exists a point whose first conjugate locus has order $n-1$ [2]. Therefore no better result is possible under these hypotheses.

REFERENCES

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