A GENERAL THEOREM FOR DECOMPOSITION OF LINEAR RANDOM PROCESSES

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Abstract. Let $E$ and $F$ be locally convex spaces in duality and let $f$ be a linear random process indexed by $F$ such that the corresponding cylindrical measure is a Radon measure. It is shown without any assumptions of metrizability or countability that there is an equivalent process with continuous linear trajectories.

1. Introduction. A fundamental theorem in the theory of generalized random processes and Radonifying maps may be stated roughly as follows: If $f$ is a linear random process indexed by the dual $F$ of a locally convex space $E$ whose compact subsets are metrizable, and if the cylinder measure induced by $f$ is a Radon measure on $E$, then there is a measurable map $g$ with values in $E$ which decomposes $f$, i.e., $\langle g(t), y \rangle = [f(y)](t)$ a.e. for each $y$ in $F$. As noted by L. Schwartz in [4, (XIII. 5)], the theory of Radonifying maps often introduces spaces in which compact sets are not metrizable. It is desirable to find a more general version of this theorem which does not assume metrizability of compact subsets. The purpose of this article is to establish such a result.

2. Preliminaries. Let $E$ and $F$ be real locally convex Hausdorff linear topological spaces in duality. Let $I$ denote the collection of all finite (ordered) linearly independent subsets of $F$. If $a$ is in $I$ then $F_a$ is the linear span of $a$, and $a \leq b$ means $F_a$ is contained in $F_b$. For each $a$ in $I$ there is a natural mapping $p_a$ of $E$ onto the dual $E_a$ of $F_a$; for $x$ in $E$, $p_a(x)$ is the restriction of $x$, considered as a linear functional, to $F_a$. If $a \leq b$ there is a similar natural map $p_{ab}$ of $E_b$ onto $E_a$. If $p_{aa}$ is the identity map on $E_a$, then since for $a \leq b \leq c$, $p_{ac} = p_{ab} \circ p_{bc}$ and since the maps $p_{ab}$ are continuous, linear, and surjective, the family $(E_a, p_{ab})$ forms a projective system of linear topological spaces. The maps $p_a$ form a consistent family of continuous linear maps since $p_a = p_{ab} \circ p_b$ for $a \leq b$. Since $F$ separates the points of $E$, the maps $p_a$ separate the points of $E$; i.e., if $x \neq y$ in $E$ then for some $a$, $p_a(x) \neq p_a(y)$.

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If \((\Omega, \mathcal{F}, \mu)\) is a finite measure space, then \(L^0 = L^0(\Omega, \mathcal{F}, \mu)\) denotes the space of \(\mu\)-equivalence classes of \(\mathcal{F}\)-measurable real valued functions. A version of a member \(l\) of \(L^0\) is an element of the equivalence class of \(l\). A linear mapping of \(F\) into \(L^0\) is called a linear process indexed by \(F\). A version of a linear process \(f\) is a linear mapping \(g\) of \(F\) into the space of all measurable functions such that for each \(y\) in \(F\), \(g(y) \in f(y)\). A version \(g\) of a linear process \(f\) has continuous linear trajectories if for almost all \(t\) in \(\Omega\), 
\[ [g(\cdot)](t) \]
is a continuous linear function on \(F\). If a version \(g\) of \(f\) has continuous linear trajectories, then the function \(h\) of \(\Omega\) into \(E\) defined by 
\[ \langle h(t), y \rangle = [g(y)](t) \]
is called a decomposition of \(f\) into \(E\).

For each \(a\) in \(I\) let \(\mathcal{B}_a\) be the Borel \(\sigma\)-field of \(E_a\), and define \(\mathcal{C}\) to be the collection of all subsets of \(E\) of the form \(p_a^{-1}(B)\) for \(B\) in \(\mathcal{B}_a\), and \(a\) in \(I\). Members of \(\mathcal{C}\) are called cylinder sets of \(E\). Let \(f\) be a linear process indexed by \(F\), mapping \(F\) into \(L^0(\Omega, \mathcal{F}, \mu)\). For each \(a\) in \(I\) there is a measurable map \(f_a\) of \(\Omega\) into \(E_a\) defined as follows: If \(a = \{y_1, \ldots, y_n\}\), let \(f_k\) be a version of \(f(y_k)\) and define \(f_a(t) = (f_1(t), \ldots, f_n(t))\) in \(E_a\). If \(a = \{y\}\) we write \(f_y\) instead of \(f_a\). If \(y\) is in \(F_a\) and \(y = \sum b_k y_k\) then 
\[ \langle f_a(t), y \rangle = \sum b_k f_k(t) = [f(y)](t) \]
a.e. (\(\mu\)). For each \(a\), the Borel measure \(\lambda_a = f_a(\mu)\) defined by \(\lambda_a(B) = \mu(f_a^{-1}(B))\) is independent of the version of \(f_a\). If \(a \leq b\) then \(p_{ab} f_a(\omega) = f_a(\omega)\) a.e. and \(p_{ab}(\lambda_a) = \lambda_a\). The finitely additive function \(\lambda\) defined on \(\mathcal{C}\) by the formula \(\lambda(p_a^{-1}(B)) = \lambda_a(B)\) is called the cylinder measure induced by \(f\). Linear processes \(f\) and \(g\) indexed by \(F\) with values in possibly different spaces \(L^0\) are said to be equivalent if they determine the same cylinder measure.

The statement that \(A\) is contained in \(B\), a.e. (\(\mu\)), means that \(I_A \subseteq I_B\), a.e. (\(\mu\)), where \(I_A\), \(I_B\) are the indicator (characteristic) functions for \(A\) and \(B\) respectively.

Let \(F^*\) denote the algebraic dual of \(F\). For each \(a\) in \(I\) there is a natural map \(\pi_a\) of \(F^*\) onto \(E_a\) defined by \(\langle \pi_a(x), y \rangle = \langle x, y \rangle\) for \(x\) in \(F^*\) and \(y\) in \(E_a\). Let \(\mathcal{C}^*\) denote the collection of cylinder sets of \(F^*\) (sets of the form \(p_a^{-1}(B)\), \(B\) in \(\mathcal{B}_a\), \(a\) in \(I\)). If \(\lambda\) is a cylinder measure on \(E\) determined by a linear process \(f\) then there is a cylinder measure \(\lambda'\) on \(F^*\) defined by \(\lambda'(p_a^{-1}(B)) = \lambda_a(B) = \mu(f_a^{-1}(B))\). By a theorem of Bochner, (cf. Badrikian [1]), the cylinder measure \(\lambda'\) extends to a (countably additive) measure on \(\mathcal{C}^*\), the \(\sigma\)-algebra generated by \(\mathcal{C}^*\). Suppose \(\lambda\) extends to a bounded Radon measure on \(E\), i.e., a finite measure on the Borel sets of \(E\) such that \(\lambda(B) = \sup(\lambda(C)\mid C\ is a\ compact\ subset\ of\ B)\) for each Borel set \(B\). The \(\lambda'\)-outer measure of \(E\) in \(F^*\) is then \(\lambda(E)\), for if \(E\) is contained in the union of a sequence of sets \(p_a^{-1}(A_n)\), where \(A_n\) is in \(\mathcal{B}_{a_n}\), then \(E\) is contained in the union of the sequence \(\{p_a^{-1}(A_n)\}\); hence

\[
\lambda(E) \leq \sum \lambda(p_a^{-1}(A_n)) = \sum \lambda'(p_a^{-1}(A_n)).
\]
3. **The shadow of a compact subset of $E$.** In this section, $T$ is a Hausdorff topological space, $\mu$ is a bounded Radon measure on $T$ and $f$ is a linear process indexed by $F$, mapping $F$ into $L^0(T, \mu)$. Suppose that $K$ is a compact subset of $E$. Let $K_a = p_a(K)$ and assume that $\inf\{\lambda_a(K_a) : a \in I\} = c > 0$, where $\lambda$ is the cylinder measure induced by $f$ and $\lambda_a = p_a(\lambda)$ as before. Restricting the maps $p_a$ and $p_{ab}$ to $K$ and $K_b$, the family $(K_a, p_{ab})$ forms a projective system of topological spaces and the maps $p_a$ form a consistent family of continuous maps which separate the points of $K$. If $\gamma_a$ is the restriction of $\lambda_a$ to $K$ then the family of measures $\gamma_a$ forms a subprojective system, i.e., $p_{ab}(\gamma_b) \leq \gamma_a$. A fundamental proposition in Bourbaki [2, p. 51] states that there is a unique bounded Radon measure $\gamma$ on $K$ such that $\gamma(C) = \inf\{\gamma_a(p_a(C)) : a \in I\}$ for each compact subset $C$ of $K$. If $\gamma$ extends to a Radon measure on $E$ then $\gamma$ is the restriction of $\lambda$ to $K$ [2, p. 52].

Consider the projective system $(T_a, i_{ab})$ where $T_a = T$ and $i_{ab}$ is the identity map. Let $M_a = f_a^{-1}(K_a)$ and let $\nu_a$ be the restriction of $\mu$ to $M_a$. Note then that $f_a(\nu_a) = \gamma_a$. If $a \leq b$ then since $f_{ab} = p_{ab} \circ f_b$, a.e. (\mu), $M_b$ is contained in $M_a$, a.e. (\mu). Since for each Borel set $A$ of $T$, $i_{ab}(\nu_b)(A) = \nu_b(A) = \mu(A \cap M_b) \leq \mu(A \cap M_a) = \nu_a(A)$, the family of measures $\nu_a$ forms a subprojective system. It follows as before that there is a unique bounded Radon measure $\nu$ on $T$ such that $\nu(C) = \inf\{\nu_a(C) : a \in I\}$ for each compact subset $C$ of $T$. For each $a$, $\nu_a$ is absolutely continuous with respect to $\mu$, hence $\nu$ is absolutely continuous with respect to $\mu$. Let $r$ be the Radon-Nikodym derivative of $\nu$ with respect to $\mu$. The support $M$ of $r$ is called the **shadow** of $K$ in $T$ ($M = \{t : r(t) \neq 0\}$). The following lemmas give the main properties of $M$.

3.1. **Lemma.** $M$ is contained in $M_a$, a.e. (\mu).

**Proof.** It suffices to show that $r = 0$, a.e. (\mu), on the complement of $M_a$. If $Q$ is the complement of $M_a$, then $\int_Q r \, d\mu = \nu Q \leq \nu_a(Q) = 0$. Since $r$ is nonnegative, $r = 0$, a.e. (\mu), on $Q$.

3.2. **Lemma.** $\mu(M) = \inf\{\mu(M_a) : a \in I\}$. Hence

$$\nu(T) = \mu(M) = \inf\{\gamma_a(K_a) : a \in I\} = \inf\{\lambda_a(K_a) : a \in I\} = c.$$

If $\gamma$ extends to a Radon measure on $E$, then $\mu(M) = \lambda(K)$.

**Proof.** For $\varepsilon > 0$ there is a compact subset $C$ of $T$ such that $\nu(C) \geq \nu(T) - \varepsilon$. For each $a$, $\mu(M_a) \geq \mu(M_a \cap C) = \nu_a(C) \geq \nu(C) \geq \nu(T) - \varepsilon$. So $c = \inf\{\mu(M_a) : a \in I\} \geq \nu(T)$. Now for $\delta > 0$ choose a compact set $D$ such that $\nu(D) \geq \nu(T) - \delta$. Since $\mu(M_a) = \lambda_a(K_a) \geq c$ for each $a$, $\nu_a(D) = \mu(D \cap M_a) \geq c - \delta$ for each $a$. Hence $\nu(D) = \inf\{\nu_a(D) : a \in A\} \geq c - \delta$. This implies $\nu(T) \geq c - \delta$. 
3.3. Lemma. If $\lambda$ extends to a bounded Radon measure on $E$, then for each $a$ in $I$ and for each $A$ in $\mathcal{B}_a$, the following properties hold:

(i) $\lambda(p_a^{-1}(A) \cap K) = \inf\{\lambda(p_a^{-1}(A) \cap p_b^{-1}(K_b)) : b \in I\}$;
(ii) $\mu(f_a^{-1}(A) \cap M) = \inf\{\mu(f_a^{-1}(A) \cap M_b) : b \in I\}$;
(iii) $\lambda(p_a^{-1}(A) \cap p_b^{-1}(K_b)) = \mu(f_a^{-1}(A) \cap M_b)$ for each $b \geq a$;
(iv) $\lambda(p_a^{-1}(A) \cap K) = \mu(f_a^{-1}(A) \cap M)$.

Proof. (i) For $e > 0$ pick $b \geq a$ such that $\lambda(p_b^{-1}(K_b)) - \lambda(K) < e$; then $\lambda(p_b^{-1}(A) \cap p_b^{-1}(K_b)) - \lambda(p_b^{-1}(A) \cap K) < e$, since $p_b^{-1}(K_b)$ contains $K$.

(ii) For $e > 0$, Lemma 3.2 provides $b \geq a$ such that $\mu(M_b) - \mu(M) < e$; then since $M_b$ contains $M$ a.e., $\mu(f_a^{-1}(A) \cap M) - \mu(f_a^{-1}(A) \cap M) < e$.

$\lambda(p_a^{-1}(A) \cap p_b^{-1}(K_b)) = \lambda(p_b^{-1} \circ p_a^{-1}(A) \cap p_b^{-1}(K_b)) = \mu(f_a^{-1}(p_b^{-1}(A) \cap K_b)) = \mu(f_a^{-1}(A) \cap M_b)$.

(iv) follows from (i), (ii), and (iii).

3.4. Lemma. The restriction of $\mu$ to $M$ is $\nu$.

Proof. If $C$ is a compact subset of $M$, then for each $a$ in $I$, $M_a$ contains $C$, a.e. ($\mu$), by Lemma 3.1. It follows that $\nu_a(C) = \mu(C \cap M_a) = \mu(C)$, hence $\nu(C) = \mu(C)$.

3.5. Lemma. If $h(y)$ is the restriction of $f(y)$ to $M$ for each $y$ in $F$, then $h$ is a linear mapping of $F$ into $L^\infty(M, \nu)$.

Proof. If $y \in F$ then $p_y(K) = K_y$ is a compact set of real numbers; hence $K_y$ is contained in an interval of the form $[-m, m]$. Let $A$ be the subset of $T$ on which $|f_y(t)| > m$ ($f_y$ is a version of $f(y)$). Since $f_y^{-1}[-m, m]$ contains $f_y^{-1}(K_y) = M_y$ which contains $M$, a.e. ($\mu$), it follows that $\nu(A) = 0$.

3.6. Lemma. There is a mapping $g$ of $M$ into $F^*$ such that $(g(t), y) = f_y(t)$, a.e. ($\nu$), for each $y$ in $F$.

Proof. By the lifting theorem of Ionescu-Tulcea [3], there is a linear mapping $\rho$ of $L^\infty(M, \nu)$ into $L^\infty(M, \nu)$, the space of all bounded measurable functions on $T$, such that $\rho(l) = h$ for each $l$ in $L^\infty(M, \nu)$. Let $(g(t), y) = [\rho \circ h(y)](t)$, where $h(y)$ is the restriction of $f(y)$ to $M$, for each $y$ in $F$.

Now suppose that $K$ and $L$ are compact subsets of $E$ such that $K$ is a subset of $L$. Let $M$ and $N$ be the shadows of $K$ and $L$ respectively in $T$. Assume also that $0 < c = \inf\{\lambda_a(K_a) : a \in I\}$ and $0 < d = \inf\{\lambda_a(L_a) : a \in I\}$.

3.7. Lemma. $M$ is contained in $N$, a.e. ($\mu$).

Proof. Let $M_a = f_a^{-1}(K_a)$ and let $N_a = f_a^{-1}(L_a)$ for each $a$. There are increasing sequences $(a_k)$ and $(b_k)$ in $I$ such that $c = \inf\{\lambda_a(K_a) : a \in I\} = \inf\{\mu(M_a)\}$ and $d = \inf\{\mu(N_a)\}$. For each $k$ choose $e_k \geq a_k$, $b_k$, $e_{k-1}$. Now
since $\mu(M_\infty) \leq \mu(M_{k})$ and $M_{k\infty}$ contains $M_{k}$, a.e. ($\mu$), $\inf\{\mu(M_{k})\} = c = \mu(\cap M_{k})$. Since $M_{a}$ contains $M$, a.e. ($\mu$), for each $a$, $\cap M_{a}$ contains $M$, a.e. ($\mu$) and since $\mu(M) = c = \mu(\cap M_{a})$, by Lemma 3.2, it follows that $M = \cap M_{k}$, a.e. ($\mu$). By similar reasoning $N = \cap N_{k}$, a.e. ($\mu$). Since for each $k$, $N_{k}$ contains $M_{k}$, a.e. ($\mu$), $N$ contains $M$, a.e. ($\mu$).

4. Statement and proof of the theorem.

Theorem. If $E$ and $F$ are locally convex spaces in duality and $f$ is a linear process mapping $F$ into $L^{0}(T, \mu)$ where $T$ is a Hausdorff space and $\mu$ is a bounded Radon measure on $T$, and if the cylinder measure induced by $f$ has an extension which is a Radon measure on $E$, then there is a subset $W$ of $T$, a sub-$\sigma$-algebra $\mathcal{F}$ of the Borel sets of $W$, a measure $\nu$ on $\mathcal{F}$, and a linear process $h$ mapping $F$ into $L^{0}(W, \mathcal{F}, \nu)$ such that $h$ is equivalent to $f$ and $h$ has a version with continuous linear trajectories.

Proof. Choose an increasing sequence $(K_{n})$ of compact subsets of $E$ such that, for each $n$, $\lambda(K_{n}) = c = \lambda(E) - 1/n$. Let $M_{k}$ be the shadow of $K_{n}$ in $T$, let $\mu_{n}$ be the restriction of $\mu$ to $M_{n}$ and let $g_{n}$ be a mapping of $M_{n}$ into $F^{*}$ such that $(g_{n}(t), y) = f_{n}(t)$, a.e. ($\mu_{n}$), as in Lemma 3.6. Let $N_{n}$ be the union of the sets $M_{k}$ for $k \leq n$ and define $g(t) = g_{n}(t)$ for $t$ in $N_{n} - N_{n-1}$. If $Y$ is the union of the sets $M_{n}$, then $(g(t), y) = f_{n}(t)$, a.e. ($\mu$), on the set $Y$. Since $\lambda(E) = \sup \lambda(K_{n}) = \sup \mu(M_{n})$ by Lemma 3.2, $\mu(Y) = \lambda(E)$. Let $\mu_{0}$ be the restriction of $\mu$ to the Borel sets of $Y$ of the form $g^{-1}(S)$ for $S$ in $\sigma\mathcal{C}^{*}$. For each Borel set $A$ of $E_{a}$,

$$\mu_{0}(g^{-1} \circ \pi_{a}^{-1}(A)) = \mu([Y \cap g^{-1} \circ \pi_{a}^{-1}(A)] = \mu([\cup (N_{n} - N_{n-1}) \cap f_{a}^{-1}(A)])$$

$$= \mu([Y \cap f_{a}^{-1}(A)]) = \sup \mu(M_{n} \cap f_{a}^{-1}(A))$$

$$= \sup \lambda(K_{n} \cap p_{a}^{-1}(A)) = \lambda(p_{a}^{-1}(A)) = \lambda'(\pi_{a}^{-1}(A)).$$

Let $W = g^{-1}(E)$. If $\mu_{0}$ is the outer measure induced by $\mu_{0}$ on subsets of $Y$ then since the $\lambda'$-outer measure of $E$ is $\lambda(E)$,

$$\mu_{0}^{*}(W) = \inf\{\mu_{0}(g^{-1}(A)) : A \in \sigma\mathcal{C}^{*}, W \subseteq g^{-1}(A)\}$$

$$= \inf\{\lambda'(A) : A \in \sigma\mathcal{C}^{*}, E \subseteq A\} = \lambda(E) = \mu_{0}(Y).$$

Let $\mathcal{F}$ be the $\sigma$-field consisting of sets of the form $g^{-1}(A) \cap W$ for $A$ in $\sigma\mathcal{C}^{*}$. Define $\nu(g^{-1}(A) \cap W) = \mu_{0}(g^{-1}(A)) = \lambda'(A)$ for each $A$ in $\sigma\mathcal{C}^{*}$; then $\nu$ is a measure on $\mathcal{F}$. If $[hy](t) = p_{0} \circ g(t) = (g(t), y)$ for each $y$ in $F$ and $t$ in $W$, then $h$ is a (version with continuous linear trajectories of $a$) linear process mapping $F$ into $L^{0}(W, \mathcal{F}, \nu)$. The cylinder measure induced by $h$ is $\lambda$ so that $h$ and $f$ are equivalent processes.

Remark. The theorem mentioned in the introduction is contained in the above. If there is a countable cofinal family of finite dimensional subspaces of $F$, in particular, if compact subsets of $E$ are metrizable, then the
compact sets $K^n$ in the preceding proof are members of $\sigma \mathcal{C}^*$ ($K^n = \bigcap \pi_n^{-1}(K_a^n)$ can be expressed as a countable intersection). It follows that $Y = g^{-1}(\bigcup K^n)$ a.e.; hence $g$ can be redefined as a version of $f$ with continuous linear trajectories.

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