PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 38, Number 2, April 1973

A GENERAL THEOREM FOR DECOMPOSITION OF LINEAR RANDOM PROCESSES

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ABSTRACT. Let E and F be locally convex spaces in duality and let f be a linear random process indexed by F such that the corresponding cylindrical measure is a Radon measure. It is shown without any assumptions of metrizability or countability that there is an equivalent process with continuous linear trajectories.

1. **Introduction.** A fundamental theorem in the theory of generalized random processes and Radonifying maps may be stated roughly as follows: If f is a linear random process indexed by the dual F of a locally convex space E whose compact subsets are metrizable, and if the cylinder measure induced by f is a Radon measure on E, then there is a measurable map g with values in E which decomposes f, i.e., $\langle g(t), y\rangle = [f(y)](t)$ a.e. for each y in F. As noted by L. Schwartz in [**4**, (XIII. 5)], the theory of Radonifying maps often introduces spaces in which compact sets are not metrizable. It is desirable to find a more general version of this theorem which does not assume metrizability of compact subsets. The purpose of this article is to establish such a result.

2. **Preliminaries.** Let E and F be real locally convex Hausdorff linear topological spaces in duality. Let I denote the collection of all finite (ordered) linearly independent subsets of F. If a is in I then F_a is the linear span of a, and $a \leqq b$ means F_a is contained in F_b. For each a in I there is a natural mapping p_a of E onto the dual E_a of F_a; for x in E, $p_a(x)$ is the restriction of x, considered as a linear functional, to F_a. If $a \leqq b$ there is a similar natural map p_{ab} of E_b onto E_a. If p_{aa} is the identity map on E_a, then since for $a \leqq b \leqq c$, $p_{ac} = p_{ab} \circ p_{bc}$ and since the maps p_{ab} are continuous, linear, and surjective, the family (E_a, p_{ab}) forms a projective system of linear topological spaces. The maps p_a form a consistent family of continuous linear maps since $p_a = p_{ab} \circ p_b$ for $a \leqq b$. Since F separates the points of E, the maps p_a separate the points of E; i.e., if $x \neq y$ in E then for some a, $p_a(x) \neq p_a(y)$.

Received by the editors February 4, 1972 and, in revised form, August 4, 1972.
AMS (MOS) subject classifications (1970). Primary 28A40, 46G15, 46A05, 46F05, 60G20.
Key words and phrases. Generalized processes, cylindrical measures, lifting, decomposition.

If $(\Omega, \mathscr{F}, \mu)$ is a finite measure space, then $L^0 = L^0(\Omega, \mathscr{F}, \mu)$ denotes the space of μ-equivalence classes of $\mathscr{F}$-measurable real valued functions. A version of a member l of L^0 is an element of the equivalence class of l. A linear mapping of F into L^0 is called a linear process indexed by F. A version of a linear process f is a linear mapping g of F into the space of all measurable functions such that for each y in F, $g(y) \in f(y)$. A version g of a linear process f has continuous linear trajectories if for almost all t in Ω, $[g(\cdot)](t)$ is a continuous linear function on F. If a version g of f has continuous linear trajectories, then the function h of Ω into E defined by $\langle h(t), y\rangle = [g(y)](t)$ is called a decomposition of f into E.

For each a in I let $\mathscr{B}_a$ be the Borel σ-field of E_a, and define $\mathscr{C}$ to be the collection of all subsets of E of the form $p_a^{-1}(B)$ for B in $\mathscr{B}_a$, and a in I. Members of $\mathscr{C}$ are called cylinder sets of E. Let f be a linear process indexed by F, mapping F into $L^0(\Omega, \mathscr{F}, \mu)$. For each a in I there is a measurable map f_a of Ω into E_a defined as follows: If $a = \{y_1, \cdots, y_n\}$, let f_k be a version of $f(y_k)$ and define $f_a(t) = (f_1(t), \cdots, f_n(t))$ in E_a. If $a = \{y\}$ we write f_y instead of f_a. If y is in F_a and $y = \sum b_k y_k$ then $\langle f_a(t), y\rangle = \sum b_k f_k(t) = [f(y)](t)$, a.e. (μ). For each a, the Borel measure $\lambda_a = f_a(\mu)$ defined by $\lambda_a(B) = \mu(f_a^{-1}(B))$ is independent of the version of f_a. If $a \leqq b$ then $p_{ab} f_b(\omega) = f_a(\omega)$ a.e. and $p_{ab}(\lambda_b) = \lambda_a$. The finitely additive function λ defined on $\mathscr{C}$ by the formula $\lambda(p_a^{-1}(B)) = \lambda_a(B)$ is called the cylinder measure induced by f. Linear processes f and g indexed by F with values in possibly different spaces L^0 are said to be equivalent if they determine the same cylinder measure.

The statement that A is contained in B, a.e. (μ), means that $I_A \leqq I_B$, a.e. (μ), where I_A, I_B are the indicator (characteristic) functions for A and B respectively.

Let F^* denote the algebraic dual of F. For each a in I there is a natural map π_a of F^* onto E_a defined by $\langle \pi_a(x), y\rangle = \langle x, y\rangle$ for x in F^* and y in F_a. Let $\mathscr{C}^*$ denote the collection of cylinder sets of F^* (sets of the form $\pi_a^{-1}(B)$, B in $\mathscr{B}_a$, a in I). If λ is a cylinder measure on E determined by a linear process f then there is a cylinder measure λ' on F^* defined by $\lambda'(\pi_a^{-1}(B)) = \lambda_a(B) = \mu(f_a^{-1}(B)) = \lambda(p_a^{-1}(B))$. By a theorem of Bochner, (cf. Badrikian [**1**]), the cylinder measure λ' extends to a (countably additive) measure on $\sigma\mathscr{C}^*$, the σ-algebra generated by $\mathscr{C}^*$. Suppose λ extends to a bounded Radon measure on E, i.e., a finite measure on the Borel sets of E such that $\lambda(B) = \sup\{\lambda(C) : C$ is a compact subset of $B\}$ for each Borel set B. The λ'-outer measure of E in F^* is then $\lambda(E)$, for if E is contained in the union of a sequence of sets $\pi_{a_n}^{-1}(A_n)$, where A_n is in $\mathscr{B}_{a_n}$, then E is contained in the union of the sequence $\{p_{a_n}^{-1}(A_n)\}$; hence

$$\lambda(E) \leqq \lambda\left(\bigcup p_{a_n}^{-1}(A_n)\right) \leqq \sum \lambda(p_{a_n}^{-1}(A_n)) = \sum \lambda'(\pi_{a_n}^{-1}(A_n)).$$

3. **The shadow of a compact subset of E.** In this section, T is a Hausdorff topological space, μ is a bounded Radon measure on T and f is a linear process indexed by F, mapping F into $L^0(T, \mu)$. Suppose that K is a compact subset of E. Let $K_a=p_a(K)$ and assume that $\inf\{\lambda_a(K_a):a\in I\}=c>0$, where λ is the cylinder measure induced by f and $\lambda_a=p_a(\lambda)$ as before. Restricting the maps p_a and p_{ab} to K and K_b, the family (K_a, p_{ab}) forms a projective system of topological spaces and the maps p_a form a consistent family of continuous maps which separate the points of K. If γ_a is the restriction of λ_a to K then the family of measures γ_a forms a subprojective system, i.e., $p_{ab}(\gamma_b)\leqq\gamma_a$. A fundamental proposition in Bourbaki [**2**, p. 51] states that there is a unique bounded Radon measure γ on K such that $\gamma(C)=\inf\{\gamma_a(p_a(C)):a\in I\}$ for each compact subset C of K. If γ extends to a Radon measure on E then γ is the restriction of λ to K [**2**, p. 52].

Consider the projective system (T_a, i_{ab}) where $T_a=T$ and i_{ab} is the identity map. Let $M_a=f_a^{-1}(K_a)$ and let ν_a be the restriction of μ to M_a. Note then that $f_a(\nu_a)=\gamma_a$. If $a\leqq b$ then since $f_a=p_{ab}\circ f_b$, a.e. (μ), M_b is contained in M_a, a.e. (μ). Since for each Borel set A of T, $i_{ab}(\nu_b)(A)=\nu_b(A)=\mu(A\cap M_b)\leqq\mu(A\cap M_a)=\nu_a(A)$, the family of measures ν_a forms a subprojective system. It follows as before that there is a unique bounded Radon measure ν on T such that $\nu(C)=\inf\{\nu_a(C):a\in I\}$ for each compact subset C of T. For each a, ν_a is absolutely continuous with respect to μ, hence ν is absolutely continuous with respect to μ. Let r be the Radon-Nikodym derivative of ν with respect to μ. The support M of r is called the *shadow* of K in T ($M=\{t:r(t)\neq 0\}$). The following lemmas give the main properties of M.

3.1. LEMMA. *M is contained in M_a, a.e. (μ).*

PROOF. It suffices to show that $r=0$, a.e. (μ), on the complement of M_a. If Q is the complement of M_a, then $\int_Q r\,d\mu=\nu Q\leqq\nu_a(Q)=0$. Since r is nonnegative, $r=0$, a.e. (μ), on Q.

3.2. LEMMA. $\mu(M)=\inf\{\mu(M_a):a\in I\}$. *Hence*

$$\nu(T)=\mu(M)=\inf\{\gamma_a(K_a):a\in I\}=\inf\{\lambda_a(K_a):a\in I\}=c.$$

If λ extends to a Radon measure on E, then $\mu(M)=\lambda(K)$.

PROOF. For $\varepsilon>0$ there is a compact subset C of T such that $\nu(C)\geqq\nu(T)-\varepsilon$. For each a, $\mu(M_a)\geqq\mu(M_a\cap C)=\nu_a(C)\geqq\nu(C)\geqq\nu(T)-\varepsilon$. So $c=\inf\{\mu(M_a):a\in I\}\geqq\nu(T)$. Now for $\delta>0$ choose a compact set D such that $\nu(D)\geqq\nu(T)-\delta$. Since $\mu(M_a)=\lambda_a(K_a)\geqq c$ for each a, $\nu_a(D)=\mu(D\cap M_a)\geqq c-\delta$ for each a. Hence $\nu(D)=\inf\{\nu_a(D)=a\in A\}\geqq c-\delta$. This implies $\nu(T)\geqq c-\delta$.

3.3. LEMMA. *If λ extends to a bounded Radon measure on E, then for each a in I and for each A in $\mathscr{B}_a$, the following properties hold*:

(i) $\lambda(p_a^{-1}(A)\cap K)=\inf\{\lambda(p_a^{-1}(A)\cap p_b^{-1}(K_b)):b\in I\}$;

(ii) $\mu(f_a^{-1}(A)\cap M)=\inf\{\mu(f_a^{-1}(A)\cap M_b):b\in I\}$;

(iii) $\lambda(p_a^{-1}(A)\cap p_b^{-1}(K_b))=\mu(f_a^{-1}(A)\cap M_b)$ *for each* $b\geqq a$;

(iv) $\lambda(p_a^{-1}(A)\cap K)=\mu(f_a^{-1}(A)\cap M)$.

PROOF. (i) For $\varepsilon>0$ pick $b\geqq a$ such that $\lambda(p_b^{-1}(K_b))-\lambda(K)<\varepsilon$; then $\lambda(p_a^{-1}(A)\cap p_b^{-1}(K_b))-\lambda(p_a^{-1}(A)\cap K)<\varepsilon$, since $p_b^{-1}(K_b)$ contains K.

(ii) For $\varepsilon>0$, Lemma 3.2 provides $b\geqq a$ such that $\mu(M_b)-\mu(M)<\varepsilon$; then since M_a contains M a.e., $\mu(f_a^{-1}(A)\cap M_b)-\mu(f_a^{-1}(A)\cap M)<\varepsilon$.

(iii)
$$\begin{aligned}\lambda(p_a^{-1}(A)\cap p_b^{-1}(K_b)) &= \lambda(p_b^{-1}\circ p_{ab}^{-1}(A)\cap p_b^{-1}(K_b))\\ &= \mu(f_b^{-1}(p_{ab}^{-1}(A)\cap K_b)) = \mu(f_a^{-1}(A)\cap M_b).\end{aligned}$$

(iv) follows from (i), (ii), and (iii).

3.4. LEMMA. *The restriction of μ to M is ν.*

PROOF. If C is a compact subset of M, then for each a in I, M_a contains C, a.e. (μ), by Lemma 3.1. It follows that $\nu_a(C)=\mu(C\cap M_a)=\mu(C)$, hence $\nu(C)=\mu(C)$.

3.5. LEMMA. *If $h(y)$ is the restriction of $f(y)$ to M for each y in F, then h is a linear mapping of F into $L^\infty(M,\nu)$.*

PROOF. If $y\in F$ then $p_y(K)=K_y$ is a compact set of real numbers; hence K_y is contained in an interval of the form $[-m,m]$. Let A be the subset of T on which $|f_y(t)|>m$ (f_y is a version of $f(y)$). Since $f_y^{-1}[-m,m]$ contains $f_y^{-1}(K_y)=M_y$ which contains M, a.e. (μ), it follows that $\nu(A)=0$.

3.6. LEMMA. *There is a mapping g of M into F^* such that $\langle g(t),y\rangle=f_y(t)$, a.e. ($\nu$), for each y in F.*

PROOF. By the lifting theorem of Ionescu-Tulcea [3], there is a linear mapping ρ of $L^\infty(M,\nu)$ into $\mathscr{L}^\infty(M,\nu)$, the space of all bounded measurable functions on T, such that $\rho(l)\in h$ for each l in $L^\infty(M,\nu)$. Let $\langle g(t),y\rangle=[\rho\circ h(y)](t)$, where $h(y)$ is the restriction of $f(y)$ to M, for each y in F.

Now suppose that K and L are compact subsets of E such that K is a subset of L. Let M and N be the shadows of K and L respectively in T. Assume also that $0<c=\inf\{\lambda_a(K_a):a\in I\}$ and $0<d=\inf\{\lambda_a(L_a):a\in I\}$.

3.7. LEMMA. *M is contained in N, a.e. (μ).*

PROOF. Let $M_a=f_a^{-1}(K_a)$ and let $N_a=f_a^{-1}(L_a)$ for each a. There are increasing sequences (a_k) and (b_k) in I such that $c=\inf\{\lambda_{a_k}(K_{a_k})\}=\inf\{\mu(M_{a_k})\}$ and $d=\inf\{\mu(N_{b_k})\}$. For each k choose $e_k\geqq a_k, b_k, e_{k-1}$. Now since $\mu(M_{a_k}) \geqq \mu(M_{e_k})$ and $M_{e_{k-1}}$ contains M_{e_k}, a.e. (μ), $\inf\{\mu(M_{e_k})\}=c=\mu(\bigcap M_{e_k})$. Since M_a contains M, a.e. (μ), for each a, $\bigcap M_{e_k}$ contains M, a.e. (μ), and since $\mu(M)=c=\mu(\bigcap M_{e_k})$, by Lemma 3.2, it follows that $M=\bigcap M_{e_k}$, a.e. (μ). By similar reasoning $N=\bigcap N_{e_k}$, a.e. (μ). Since for each k, N_{e_k} contains M_{e_k}, a.e. (μ), N contains M, a.e. (μ).

4. Statement and proof of the theorem.

THEOREM. *If E and F are locally convex spaces in duality and f is a linear process mapping F into $L^0(T, \mu)$ where T is a Hausdorff space and μ is a bounded Radon measure on T, and if the cylinder measure induced by f has an extension which is a Radon measure on E, then there is a subset W of T, a sub-σ-algebra $\mathscr{F}$ of the Borel sets of W, a measure ν on $\mathscr{F}$, and a linear process h mapping F into $L^0(W, \mathscr{F}, \nu)$ such that h is equivalent to f and h has a version with continuous linear trajectories.*

PROOF. Choose an increasing sequence (K^n) of compact subsets of E such that, for each n, $\lambda(K^n)=c_n \geqq \lambda(E)-1/n$. Let M^n be the shadow of K^n in T, let μ^n be the restriction of μ to M^n and let g^n be a mapping of M^n into F^* such that $\langle g^n(t), y\rangle = f_y(t)$, a.e. (μ^n), as in Lemma 3.6. Let N^n be the union of the sets M^k for $k \leqq n$ and define $g(t)=g^n(t)$ for t in $N^n - N^{n-1}$. If Y is the union of the sets M^n, then $\langle g(t), y\rangle = f_y(t)$, a.e. (μ), on the set Y. Since $\lambda(E)=\sup \lambda(K^n) = \sup \mu(M^n)$ by Lemma 3.2, $\mu(Y)=\lambda(E)$. Let μ_0 be the restriction of μ to the Borel sets of Y of the form $g^{-1}(S)$ for S in $\sigma\mathscr{C}^*$. For each Borel set A of E_a,

$$\begin{aligned}\mu_0(g^{-1} \circ \pi_a^{-1}(A)) &= \mu[Y \cap g^{-1} \circ \pi^{-1}(A)] = \mu[\bigcup (N^n - N^{n-1}) \cap f_a^{-1}(A)] \\ &= \mu[Y \cap f_a^{-1}(A)] = \sup \mu(M^n \cap f_a^{-1}(A)) \\ &= \sup \lambda(K^n \cap p_a^{-1}(A)) = \lambda(p_a^{-1}(A)) = \lambda'(\pi_a^{-1}(A)).\end{aligned}$$

Let $W=g^{-1}(E)$. If μ_0^* is the outer measure induced by μ_0 on subsets of Y then since the λ'-outer measure of E is $\lambda(E)$,

$$\begin{aligned}\mu_0^*(W) &= \inf\{\mu_0(g^{-1}(A)) : A \in \sigma\mathscr{C}^*,\ W \subseteq g^{-1}(A)\} \\ &= \inf\{\lambda'(A) : A \in \sigma\mathscr{C}^*,\ E \subseteq A\} = \lambda(E) = \mu_0(Y).\end{aligned}$$

Let $\mathscr{F}$ be the σ-field consisting of sets of the form $g^{-1}(A) \cap W$ for A in $\sigma\mathscr{C}^*$. Define $\nu(g^{-1}(A) \cap W) = \mu_0(g^{-1}(A)) = \lambda'(A)$ for each A in $\sigma\mathscr{C}^*$; then ν is a measure on $\mathscr{F}$. If $[hy](t) = p_y \circ g(t) = \langle g(t), y\rangle$ for each y in F and t in W, then h is a (version with continuous linear trajectories of a) linear process mapping F into $L^0(W, \mathscr{F}, \nu)$. The cylinder measure induced by h is λ so that h and f are equivalent processes.

REMARK. The theorem mentioned in the introduction is contained in the above. If there is a countable cofinal family of finite dimensional subspaces of F, in particular, if compact subsets of E are metrizable, then the compact sets K^n in the preceding proof are members of $\sigma\mathscr{C}^*$ ($K^n = \bigcap \pi_a^{-1}(K_a^n)$ can be expressed as a countable intersection). It follows that $Y = g^{-1}(\bigcup K^n)$ a.e.; hence g can be redefined as a version of f with continuous linear trajectories.

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