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THE CONGRUENCE LEMMA FOR $[y^p z]$

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ABSTRACT. A sufficient condition for membership in $[y^p z]$, based on the weight and signature of a term, is given.

In the appendix of his book [13], Ritt suggests the need to study some special differential ideals. One line of investigation has been the study of principal differential ideals $[M]$ generated by a monomial M of zero weight. This work was started by Levi [2], with later contributions by Hillman [1]; Mead [1], [3], [4], and [5]; E. S. O'Keefe [1] and [8]; K. B. O'Keefe [1], [8], [9], [10], [11], and [12]; and the author [7].

In [7], the author investigates $[y^p z]$ in the differential ring $Q\{y, z\}$. In that paper, α-terms are defined and a function $w(p, d, e)$ is found which gives the minimum weight for an α-term of signature (d, e) for any fixed $p \geqq 2$. It is then shown for $p=2, 3$, that no linear combination of α-terms is in $[y^p z]$ and that $w(p, d, e)$ is the critical weight function for $[y^p z]$. Although this is only done for two special cases in [7], it is the author's conjecture that it is true for all $p \geqq 2$. One way to prove this conjecture is to prove lemmas corresponding to Levi's Lemmas 1.1, 1.2, and 1.3 in [2]. It is the first of these lemmas, the congruence lemma, which we consider in this paper. As a result of this lemma, we are able to give a sufficient condition for membership in $[y^p z]$.

The notation and terminology in this paper is that of Levi [2]. We now recall some definitions from [7] and introduce some new ones which will be needed.

Let $A = y^p z$ for some fixed integer $p \geqq 2$. Then the kth derivative of A is

$$A_k = \sum c(i_1, \cdots, i_p, j) y_{i_1} \cdots y_{i_p} z_j$$

where the sum is over all choices of $i_1, \cdots, i_p, j$ such that $0 \leqq i_1 \leqq \cdots \leqq i_p$, $0 \leqq j$ and $j + \sum_{n=1}^{p} i_n = k$. The coefficients $c(i_1, \cdots, i_p, j)$ are integers which can be found by Leibniz' rule, but it will suffice to note that the coefficient

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of $y_{i-1}^s y_i^{p-s} z_j$ is

$$\binom{p}{s}\frac{(ip-s+j)!}{[(i-1)!]^s (i!)^{p-s} j!} = C(p, s, i, j).$$

In writing a p.p. (i.e., a power product)

$$E = y_0^{a_0} y_1^{a_1} \cdots y_m^{a_m} z_{j_1} z_{j_2} \cdots z_{j_e}, \tag{1}$$

it will be assumed that $j_1 \leqq j_2 \leqq \cdots \leqq j_e$. Let $d = \sum_{i=0}^m a_i$. Then (d, e) is called the *signature* of E, and $w = \sum_{i=1}^m i a_i + \sum_{n=1}^e j_n$ is called the *weight* of E.

We now define a *β-factor* of E relative to $[y^p z]$.

(i) If $i < e$ and $a_i \geqq p$, then $y_i^p z_{j_{i+1}}$ is a β-factor.

(ii) If $i < e$, $0 < a_{i-1} < p$, $a_{i-1} + a_i \geqq p$ and $j_{i+1} - j_i = r \leqq \min\{p-2, a_{i-1}+a_i-p\}$, then, letting $s = p + r - a_{i-1}$, $y_{i-1}^{a_{i-1}} y_i^s z_{j_{i+1}}$ is a β-factor if $s < p$.

In case (ii), r is called the *order* of the β-factor. Also, if $a_i - s \geqq t > 0$, then $y_{i-1}^{a_{i-1}} y_i^{s+t} z_{j_{i+1}}$ is called a β-factor of *excess t*.

E is said to be a *β-term* if it has one or more β-factors. If E is not a β-term, it will be called an *α-term*.

We next define a partial ordering on the p.p. Let E be the p.p. of (1) and let $E' = y_0^{a_0'} \cdots y_n^{a_n'} z_{j_1'} \cdots z_{j_e'}$. We compare E and E' by looking at

$$j_1' - j_1,\quad a_0 - a_0',\quad j_2' - j_2,\quad a_1 - a_1', \cdots, j_e' - j_e,\quad a_{e-1} - a_{e-1}'. \tag{2}$$

(It can be assumed, without loss of generality, that $n \geqq e$.) If the first nonzero difference in this sequence is positive, we say E is *higher* than E' ($E \gg E'$) or E' is *lower* than E ($E' \ll E$). If $E \gg E'$ and the first nonzero difference in (2) is $a_i - a_i'$, we say that *$E \gg E'$ at i*; and if the first nonzero difference is before $a_i - a_i'$, we say *$E \gg E'$ before i*.

We are now ready to prove the following

CONGRUENCE LEMMA. *For each fixed integer $p \geqq 2$, any β-term is congruent modulo $[y^p z]$ to a linear combination (with rational coefficients) of α-terms of the same weight and signature.*

PROOF. It suffices to show that if E is a β-term, then $E \equiv \sum c_i E_i$ where each $E_i \gg E$, the E_i have the same weight and signature as E, and the c_i are rational numbers. (It will be clear as we proceed that weight and signature are preserved at each step.)

If E has a β-factor of type (i), we need only solve the expression for $A_{pi+j_{i+1}}$ for that factor and substitute by Levi's method [2, p. 534]. All the new terms which appear will be higher than E. If E has a β-factor of type (ii) and order zero, the same kind of substitution used above again gives E as a sum of higher terms. We now proceed by induction on the order.

Suppose the congruence lemma is true for $r \leqq R$ and let E have a β-factor of order $r=R+1$. To simplify the notation, let $a_{i-1}=a$ and $j_i=n$. Then $E=E'y_{i-1}^a y_i^s z_{n+r}$. We can now write

$$E = E' \sum_{k=0}^{r} u_k y_{i-1}^k y_i^{r-k} (y_{i-1}^{a-k} y_i^{p-a+k} z_{n+r}) \tag{3}$$

where the u_k's are rational numbers (to be determined later) such that $\sum_{k=0}^{r} u_k=1$. We now use Levi's method to substitute for the factor in parentheses in each term on the right-hand side of (3). This gives us the expression

$$E \equiv E' \sum d(i_1, \cdots, i_{p+r}, j) y_{i_1} \cdots y_{i_{p+r}} z_j, \tag{4}$$

where we assume $i_1 \leqq \cdots \leqq i_{p+r}$ and the d's are rational numbers.

The terms on the right side of (4) are divided into three classes. The first class consists of all those terms with $i_1 < i-1$, $i-1=i_1=\cdots=i_q$ for $q>a$, or $j<n$. These terms are higher than E.

The second class consists of terms with $i-1=i_1=\cdots=i_{a-m}<i_{a-m+1}=\cdots=i_{p+r}=i$ and $j=n+r-m$ where $m=1, 2, \cdots, r$. The coefficient for one of these terms is

$$\sum_{k=0}^{r} u_k \frac{C(p, a-k-m, i, n+r-m)}{C(p, a-k, i, n+r)},$$

and we will choose the u_k's so that each of these sums is zero. To show that such a choice is possible, we need only show that the determinant, B, for the system of equations

$$\sum_{k=0}^{r} u_k = 1,$$

$$\sum_{k=0}^{r} u_k \frac{C(p, a-k-m, i, n+r-m)}{C(p, a-k, i, n+r)} = 0, \qquad m = 1, 2, \cdots, r,$$

is not zero. But $B=|b_{u,v}|$ where $b_{1,v}=1$ and

$$b_{u,v} = \binom{p}{a-u-v+2} \Big/ \binom{p}{a-v+1} \cdot \frac{(n+r)(n+r-1)\cdots(n+r-u+2)}{i^{u-1}} \quad \text{for } u \geqq 2.$$

By factoring out the common factors in the rows and columns of B, we are led to the determinant $C=|c_{u,v}|$, where $c_{u,v}=\binom{p}{a-u-v+2}$. The determinant C can be easily shown to be nonzero [**6**, p. 682].

The third class of terms consists of those with $i-1=i_1=\cdots=i_u<i_{u+1}=\cdots=i_{u+v}=i<i_{u+v+1}\leqq\cdots\leqq i_{p+r}$ and $j=n+r-m$ where $a-m<u<a$, $u+v<p+r$ and $2\leqq m\leqq r$. These terms are lower than E, but can be made higher than E by the following lemma.

LEMMA 1. *Suppose the congruence lemma is true for β-factors of order $r\leqq R$. If E has a β-factor of type* (ii), *order r and excess t such that $t+r\leqq R$, then $E\equiv\sum f_k E_k$* (mod$[y^p z]$) *where $E_k\gg y_{i-1}^t E$, E_k has the same weight and signature as E, and the f_k are rational numbers.*

PROOF. Let $E=E'(y_{i-1}^{a_{i-1}}y_i^{s+t}z_{j_i+r})$. By hypothesis,

$$E\equiv\sum f_k E_k\,(\mathrm{mod}[y^p z]),$$

where $E_k\gg E$. If $y_{i-1}^t E\gg E_k\gg E$, then $E_k=E'(y_{i-1}^a y_i^b y_{i+1}^{c_1}\cdots y_{i+q}^{c_q}z_{j_i+r'})$ where $a_{i-1}+t\geqq a>a_{i-1}$ or $a=a_{i-1}$ and $r'<r$. If $a\geqq p$, then $E_k\equiv\sum f_k^* E_k^*$ where $E_k^*\gg E_k$ before $i-1$, hence $E_k^*\gg y_{i-1}^t E$. If $a<p$, we assume (without loss of generality) that $j_i+r'\leqq j_{i+2}$. We observe that $a+b-p\geqq r'$, hence $y_{i-1}^a y_i^b z_{j_i+r'}$ contains a β-factor; and if it is of type (ii), it has order $r'\leqq R$. We can therefore repeat the process, which suffices to prove the lemma.

We now complete the proof of the congruence lemma by observing that each term in the third class is a β-term of type (ii), order $r'=r-m<R$, excess $t=(a_{i-1}-u)$, and $t+r'\leqq R$.

We recall from [7] that $w(p,d,e)$, the minimum weight for an α-term relative to $[y^p z]$ and having signature (d,e), is given by the following. Let $q=[d/(p-1)]$ and $r=d-q(p-1)$. Then

$$\begin{aligned} w(p,d,e) &= (p-1)(q-1)q+2qr && \text{if } e\geqq 2q,\\ &= de-(p-1)e/2-(p-1)e^2/4 && \text{if } e \text{ is even and } e<2q,\\ &= de-(p-1)e/2-(p-1)(e^2+1)/4 && \\ & && \text{if } e \text{ is odd and } e<2q. \end{aligned}$$

In view of the congruence lemma, we can now give the following sufficient condition for membership in $[y^p z]$.

THEOREM. *Let E be a p.p. in y and z. If E has a factor E' of weight w' and signature (d',e') with $w'<w(p,d',e')$, then $E\in[y^p z]$.*

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