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ISOTOPY EQUIVALENCE CLASSES OF NORMAL ARCS IN $F \times I$

C. D. FEUSTEL

ABSTRACT. Let F be a compact 2-manifold and I the closed unit interval. Let α and β be arcs embedded in $F\times I$ such that α and β meet the boundary of $F\times I$ in the boundary of α and β respectively. Then we give necessary and sufficient conditions for the existence of an ambient isotopy, constant on the boundary of $F\times I$, moving α to β. We also obtain ambient isotopies of families of arcs properly embedded in $F\times I$.

I. **Introduction.** In [4] Martin shows that two arcs having common endpoints embedded in the interior of a 3-manifold M are isotopic if and only if they are homotopic. In this paper we shall concern ourselves with the existence of ambient isotopies moving one properly embedded arc in M to another. We shall restrict ourselves to the case when our 3-manifold M is the product of a compact surface F with the unit interval I.

All spaces in this paper are simplicial complexes and all maps are piecewise linear.

Before proceeding it will be useful to establish some notation. Let X be a triangulated n-manifold. Then $\mathrm{bd}(X)$ is the $(n-1)$-submanifold of X consisting of all $(n-1)$-simplexes which are faces of exactly one n-simplex. Let Y be a manifold. An embedding X in Y is *proper* if $X\cap \mathrm{bd}(Y)=\mathrm{bd}(X)$. Throughout this paper F will be a compact connected surface and I will be the unit interval. We let $p:F\times I\to F$ be the natural projection map.

Let $\{\alpha_i:i=1,\cdots,n\}$ be a collection of disjoint arcs properly embedded in $F\times I$ such that $\mathrm{bd}(\alpha_i)\subset(F-\mathrm{bd}(F))\times\{0,1\}$ and $p\,\mathrm{bd}(\alpha_i)=\{x_i\}$ for $i=1,\cdots,n$. Then we shall say that $\{\alpha_i:i=1,\cdots,n\}$ is a *normal collection of arcs*. An *isotopy of a homeomorphism* $h:X\to Y$ is a map $H:X\times I\to Y$ such that for $h_t=H|X\times\{t\}$ we have $h_0=h$ and h_t is a homeomorphism onto Y. An *isotopy of subspaces* Z_1 and Z_2 in X is an isotopy of the identity map on X such that $h_1(Z_1)=Z_2$.

Let $\{\alpha_i:i=1,\cdots,n\}$ and $\{\beta_i:i=1,\cdots,n\}$ be normal collections of arcs in $F\times I$. Then $\{\alpha_i:i=1,\cdots,n\}$ is *homeomorphically unknotted* if there is a

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homeomorphism $h: F\times I \to F\times I$ such that

(1) $h(\alpha_i)=\{x_i\}\times I$ for $i=1, \cdots, n$,

(2) $h|F\times\{0\}\cup \mathrm{bd}(F)\times I$ is the identity.

The collection $\{\alpha_i : i=1, \cdots, n\}$ is *isotopically unknotted* if the $\bigcup_{i=1}^n \alpha_i$ and $\bigcup_{i=1}^n \{x_i\}\times I$ are isotopic under a deformation constant on $\mathrm{bd}(F\times I)$. We shall say that the collections $\{\alpha_i : i=1, \cdots, n\}$ and $\{\beta_i : i=1, \cdots, n\}$ are *homeomorphically equivalent* if there is a homeomorphism $h: F\times I \to F\times I$ such that

(1) $h(\alpha_i)=\beta_i$ for $i=1, \cdots, n$,

(2) $h|F\times\{0\}\cup \mathrm{bd}(F)\times I$ is the identity.

We shall say that the collections $\{\alpha_i : i=1, \cdots, n\}$ and $\{\beta_i : i=1, \cdots, n\}$ are *isotopically equivalent* if there is an isotopy of the subspaces $\bigcup_{i=1}^n \alpha_i$ and $\bigcup_{i=1}^n \beta_i$ which is constant on $\mathrm{bd}(F\times I)$.

A 3-manifold M is *irreducible* if every 2-sphere embedded in M bounds a 3-ball embedded in M. It is well known that $F\times I$ is irreducible if F is not the 2-sphere.

We shall try to follow Waldhausen's principal of "induction on niceness". (See [**6**, p. 58].) That is, after we convince ourselves there is no obstruction to achieving some niceness, we take up our problem again assuming that niceness.

II. **Useful results.** In [**3**] we prove the following four theorems.

THEOREM 2.1. *Let $h: F\times I\to F\times I$ be a homeomorphism such that $h|\mathrm{bd}(F)\times I\cup F\times\{0\}$ is the identity. Then there is an isotopy, constant on $\mathrm{bd}(F)\times I\cup F\times\{0\}$ of h to the identity.*

Let α be a normal arc embedded in $F\times I$. Then α is *monotonic in $F\times I$* if $\alpha\cap F\times\{t\}$ is a single point for t in I.

THEOREM 2.2. *We assume that F is not the 2-sphere. Let α be a normal arc in $M=F\times I$ such that $p\,\mathrm{bd}(\alpha)=\{x\}$. Then there is an isotopy of M constant on $\mathrm{bd}(M)$ carrying α to a monotonic arc if and only if α is homeomorphically unknotted.*

THEOREM 2.3. *Let α be a normal arc in $M=F\times I$ such that $p\,\mathrm{bd}(\alpha)=\{x\}$. Then α is isotopically unknotted if and only if α is homeomorphically unknotted and the loop $p(\alpha)$ is homotopic to a point.*

THEOREM 2.4. *Let F be a surface other than the 2-sphere. Let $F=F_1\cup F_2$ and $F_1\cap F_2=c$ be a component of $\mathrm{bd}(F_1)$. Let α be a normal arc in $F_1\times I$. Then α is a homeomorphically unknotted in $F_1\times I$ if and only if it is homeomorphically unknotted in $F\times I$.*

The following theorem is an immediate consequence of Theorem 5.2 in [**1**] and Lemma 3.4 in [**3**].

THEOREM 2.5. *Let F be a surface other than the projective plane, Klein bottle, torus, or 2-sphere. Let A and B be annuli properly embedded in $M=F\times I$ such that* $\mathrm{bd}(A)=\mathrm{bd}(B)$ *and* $\mathrm{bd}(A)\cap F\times\{j\}$ *is a non-nullhomotopic loop embedded in the interior of $F\times\{j\}$ for $j=0, 1$. Then there is an isotopy constant on* $\mathrm{bd}(F\times I)$, *carrying A to B.*

III. Equivalence of knotted arcs.

THEOREM 3.1. *Let $\{\alpha_i:i=1,\cdots,n\}$ and $\{\beta_i:i=1,\cdots,n\}$ be normal collections of arcs in $M=F\times I$ which are homeomorphically equivalent. Suppose for some t_0 where $0<t_0<1$, we have that $\alpha_i\cap F\times[t_0,1]=\beta_i\cap F\times[t_0,1]=\{x_i\}\times[t_0,1]$ for $i=1,\cdots,n$. Then there is an isotopy of M constant on* $\mathrm{bd}(M)$ *carrying $\alpha_i\cap F\times[0,t_0]$ to $\beta_i\cap F\times[0,t_0]$ and carrying $\alpha_i\cap F\times[t_0,1]$ to a monotonic arc for $i=1,\cdots,n$.*

PROOF. By assumption there is a homeomorphism $h:F\times I\rightarrow F\times I$ carrying $\{\alpha_i:i=1,\cdots,n\}$ to $\{\beta_i:i=1,\cdots,n\}$ which leaves $\mathrm{bd}(F)\times I\cup F\times\{0\}$ fixed. By Theorem 2.1 there is an isotopy of h to the identity which is constant on $\mathrm{bd}(F)\times I\cup F\times\{0\}$. Define $k:F\rightarrow F$ by $k(v)=ph(v,1)$ for v in F. Then it can be seen that there is an isotopy $K:F\times I\rightarrow F$ which is constant on $\mathrm{bd}(F)$ such that k_0 is the identity, $k_1=k^{-1}$, and K is induced by our isotopy of h. Furthermore if we define $h_1:F\times I\rightarrow F\times I$ by

(1) $h_1(v,t)=(K(v,(t-t_0)/(1-t_0)),t)$ for v in F and t in $[t_0,1]$,

(2) $h_1(v,t)=(v,t)$ for v in F and t in $[0,t_0]$,

it is easily seen that the homeomorphism $h_1h|\mathrm{bd}(F\times I)$ is the identity and h_1h is isotopic to the identity under a deformation constant on $\mathrm{bd}(F\times I)$ and level preserving on $F\times[t_0,1]$. Since $h_1h(\alpha_i\cap[0,t_0])=\beta_i\cap[0,t_0]$ and $h_1h(\alpha_i\cap[t_0,1])$ is monotonic for $i=1,\cdots,n$, this completes the proof of Theorem 3.1.

THEOREM 3.2. *Let α and β be arcs normal in $F\times I$ with $\alpha\cup\beta$ a loop. Then α and β are isotopically equivalent if and only if they are homeomorphically equivalent and the loop $\alpha\cup\beta$ is nullhomotopic in $F\times I$.*

PROOF. We need only show that α and β are isotopically equivalent if they are homeomorphically equivalent. We first find a homeomorphism $h:F\times I\rightarrow F\times I$ so that $h(\beta)\cap F\times[t_1,1]=\{x\}\times[t_1,1]$ for $0<t_1<1$ and t_1 near 1. We may denote $h(\beta)$ by β and $h(\alpha)$ by α. After a deformation constant on $\mathrm{bd}(M)$, we may assume that $\alpha\cap F\times[t_2,1]=\{x\}\times[t_2,1]$ for $0<t_2<1$ and t_2 sufficiently near 1. Let $t_0=$maximum of t_1 and t_2. By Theorem 3.1 after a deformation constant on $\mathrm{bd}(M)$, we may assume that $\alpha\cap[0,t_0]=\beta\cap[0,t_0]$ and $\alpha\cap[t_0,1]$ is a monotonic arc in $F\times[t_0,1]$. We recall that $\beta\cap[t_0,1]=\{x\}\times[t_0,1]$. Now by Theorem 2.2, $\alpha\cap[t_0,1]$ is homeomorphically unknotted since it is a monotonic arc. Since the loop

$\alpha\cup\beta$ is homotopic to a point and $\alpha\cap F\times[0, t_0]=\beta\cap F\times[0, t_0]$, the loop $(\alpha\cap F\times[t_0, 1])\cup(\beta\cap F\times[t_0, 1])$ is homotopic to a point. Since $\beta\cap F\times[t_0, 1]=\{x\}\times[t_0, 1]$, the loop $p(\alpha\cap F\times[t_0, 1])$ is homotopic to a point. By Theorem 2.3, there is an isotopy of $F\times[t_0, 1]$ constant on $\text{bd}(F\times[t_0, 1])$ carrying $\alpha\cap F\times[t_0, 1]$ to $\beta\cap F\times[t_0, 1]$ and Theorem 3.2 follows.

LEMMA 3.3. *Let α_i, $i=1, \cdots, n$, be a normal collection of arcs embedded in $M=F\times I$. Let S be a 2-sphere embedded in $M-\bigcup_{i=1}^n \alpha_i$. Then S bounds a 3-ball embedded in $M-\bigcup_{i=1}^n \alpha_i$.*

PROOF. Since M is irreducible, S bounds a 3-ball embedded in M. Since $\bigcup_{i=1}^n \alpha_i$ does not meet S, $\bigcup_{i=1}^n \alpha_i$ does not meet the interior of the ball bounded by S. Lemma 3.3 follows.

LEMMA 3.4. *Let α_i, $i=1, \cdots, n$, be a normal collection of arcs embedded in $M=F\times I$. Let A_1 and A_2 be annuli properly embedded in $M-\bigcup_{i=1}^n \alpha_i$ such that A_1 and A_2 each meet both $F\times\{0\}$ and $F\times\{1\}$ in a non-nullhomotopic orientable, simple loop and $A_1\cap F\times\{0\}=A_2\cap F\times\{0\}$. Then there is an isotopy of M carrying A_1 to A_2 which is constant on $F\times\{0\}\cup \text{bd}(F)\times I\cup\bigcup_{i=1}^n \alpha_i$.*

The proof of Lemma 3.4 is exactly what one would expect. One pushes A_1 off of A_2 except for $A_1\cap A_2\cap F\times\{0\}$. Then one pushes A_1 to A_2. It is impossible to follow Waldhausen's proof of 3.4 in **[6]** since the arcs α_i may get in the way.

PROOF. It is a consequence of Theorem 5.1 in **[1]** that we may assume that $A_2=\lambda\times I$ where λ is a simple non-nullhomotopic loop in F. After a deformation, we may assume that A_1 meets $\lambda\times I$ in a collection of disjoint simple loops embedded in $\lambda\times I$ together with a number of simple arcs with their boundary contained in $\lambda\times\{0, 1\}$.

If there is a simple nullhomotopic loop in $A_1\cap\lambda\times I$, it bounds a disk both on $\lambda\times I$ and on A_1. Thus we could find an "innermost" disk on A_1 bounded by a loop l in $A_1\cap\lambda\times I$. Then l would also bound a disk on $\lambda\times I$ and the union of these two disks would be a sphere S embedded in $M-\bigcup_{i=1}^n \alpha_i$. By Lemma 3.3, S bounds a ball embedded in $M-\bigcup_{i=1}^n \alpha_i$. Thus by the usual argument after a deformation constant on $\text{bd}(M)\cup\bigcup_{i=1}^n \alpha_i$, we may assume that $A_1\cap\lambda\times I$ contains no simple nullhomotopic loops and thus no arcs with both endpoints on $\lambda\times\{0\}$.

If $A_1\cap\lambda\times I$ contains a simple arc β with both endpoints on $\lambda\times\{1\}$, β together with a segment of $A_1\cap F\times\{1\}$ bounds a disk $\mathcal{D}_1$ on A_1. We may assume that $\mathcal{D}_1\cap\lambda\times I=\beta$. Now β together with a segment of $\lambda\times\{1\}$ bounds a disk $\mathcal{D}_2$ on $\lambda\times I$. Now $\mathcal{D}_1\cup\mathcal{D}_2$ is a disk properly embedded in M. Since $\text{bd}(\mathcal{D}_1\cup\mathcal{D}_2)$ lies on $F\times\{1\}$ and $F\times\{1\}$ is incompressible in M, $\text{bd}(\mathcal{D}_1\cup\mathcal{D}_2)$ bounds a disk $\mathcal{D}$ embedded in $F\times\{1\}$. Thus $\mathcal{D}\cup\mathcal{D}_1\cup\mathcal{D}_2$ is a 2-sphere embedded in M. By the usual argument after a deformation constant on $\bigcup_{i=1}^{n}\alpha_i\cup \mathrm{bd}(F)\times I\cup F\times\{0\}$, we may assume that $A_1\cap\lambda\times I$ contains no simple arcs running from $\lambda\times\{1\}$ to itself. Thus $A_1\cap\lambda\times I$ contains at most a single arc from $\lambda\times\{0\}$ to $\lambda\times\{1\}$. Since λ is two-sided in F, $A_i\cap\lambda\times I$ consists of a collection of simple non-nullhomotopic loops.

Let l be the loop on A_1 which bounds a subannulus A of A_1 which meets $\lambda\times I$ in l. Let $M^*=F_1\times I$ be the manifold obtained by splitting M along $\lambda\times I$ and $P:M^*\to M$ the natural projection. Let A^* be the annular component of $P^{-1}(A)$. Since $H_2(F_1\times I, F_1\times\{1\}\cup \mathrm{bd}(F_1\times I); Z_2)=0$, A^* bounds a finite 3-chain in $C_3(F_1\times I, F_1\times\{1\}\cup \mathrm{bd}\, F_1\times I; Z_2)$. Denote the 3-manifold representing one such chain by X_1 and $\mathrm{cl}(M^*-X_1)$ by X_2. We assume X_2 contains $F_1\times\{0\}$. Then $X_1\cap X_2=A^*$. Thus Van Kampen's theorem promises the diagram of groups shown in Figure 1 where all maps shown are induced by inclusion.

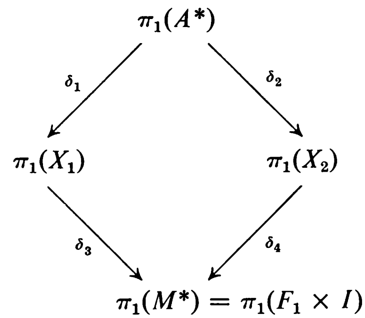

FIGURE 1

Note that δ_1 and δ_2 are monomorphisms, and δ_4 is onto since X_2 contains $F_1\times\{0\}$. It follows from 4.2 in **[2]** that δ_1 is onto. Thus by 3.1 in **[1]** there is a homeomorphism $h:X_1\to A^*\times I$ carrying A^* to $A^*\times\{0\}$. Thus it is not difficult to push A_1 across $P(X_1)$ to reduce the number of loops in $A_1\cap\lambda\times I$ since $P(X_1)$ does not meet $\bigcup_{i=1}^{n}\alpha_i$. If there is only one such loop, we could push A_1 across $P(X_1)$ onto $\lambda\times I$. This completes the proof of Lemma 3.4.

THEOREM 3.5. *Let F be a surface other than the torus, projective plane or 2-sphere and $\{\alpha_i:i=1,\cdots,n\}$ and $\{\beta_i:i=1,\cdots,n\}$ normal collections of arcs embedded in $M=F\times I$ which are homeomorphically equivalent. Let $p\,\mathrm{bd}(\alpha_i)=p\,\mathrm{bd}(\beta_i)=\{x_i\}$ for $i=1,\cdots,n$. Let $\lambda_1,\cdots,\lambda_m$ be a collection of disjoint, non-nullhomotopic, simple, orientable loops properly embedded in $F-\{x_i:i=1,\cdots,n\}$ such that each component of $F-\bigcup_{j=1}^{m}\lambda_j$ contains at most one of the x_i. Assume that $\lambda_j\times\{0\}$ is freely homotopic to $\lambda_j\times\{1\}$ in $M-\bigcup_{i=1}^{n}\alpha_i$ and in $M-\bigcup_{i=1}^{n}\beta_i$. If the loop $\alpha_i\cup\beta_i$ is nullhomotopic for $i=1,\cdots,n$, then the collections $\{\alpha_i:i=1,\cdots,n\}$ and $\{\beta_i:i=1,\cdots,n\}$ are isotopically equivalent.*

PROOF. The proof of this theorem is almost a triviality since we have Theorem 3.2 and Lemma 3.4. We simply arrange to have each pair of normal arcs lie in its own product manifold and then apply Theorem 3.2.

It is a consequence of the generalization of Dehn's lemma in [5] that $\lambda_j\times\{0, 1\}$ bounds annuli A_j in $F\times I-\bigcup_{i=1}^{n}\alpha_i$ and B_j in $F\times I-\bigcup_{i=1}^{n}\beta_i$ for $j=1,\cdots,m$. Using standard techniques we may assume that the A_j are pairwise disjoint as are the B_j. It is a consequence of Theorem 5.2 in [1] that after a homeomorphism we may assume $B_j=\lambda_j\times I$ for $j=1,\cdots,k$. It is a consequence of Theorem 2.5 that, after a deformation constant on bd(M), we may assume $A_j=\lambda_j\times I$ for $j=1,\cdots,m$. Let $h:M\to M$ be a homeomorphism such that $h(\alpha_i)=\beta_i$ for $i=1,\cdots,n$, and $h|F\times\{0\}\cup$ bd$(F)\times I=$id. If $h|\bigcup_{j=1}^{m}\lambda_j\times I=$id, the theorem would follow from 3.2. It is a consequence of Lemma 3.4 that we may assume that $h(\lambda_j\times I)=\lambda_j\times I$ after a deformation constant on $\bigcup_{j=1}^{n}\beta_j\cup F\times\{0\}\cup$bd$(F)\times I$. After a deformation constant outside of a small neighborhood of $\bigcup_{j=1}^{m}\lambda_j\times I$, we can assume $h|\lambda_j\times I=$id for $j=1,\cdots,m$. This completes the proof of Theorem 3.5.

We will denote the *Euler characteristic* of a manifold F by $\chi(F)$.

THEOREM 3.6. *Let* $\{\alpha_i:i=1,\cdots,n\}$ *and* $\{\beta_i:i=1,\cdots,n\}$ *be normal collections of arcs, embedded in* $M=F\times I$, *which are homeomorphically equivalent. We suppose that* $\chi(F)<0$ *and that for every simple loop* λ *in* $F-\{x_i:i=1,\cdots,n\}$, $\lambda\times\{0\}$ *is freely homotopic to* $\lambda\times\{1\}$ *in* $M-\bigcup_{i=1}^{n}\alpha_i$ *and in* $M-\bigcup_{i=1}^{n}\beta_i$. *Then the collections* $\{\alpha_i:i=1,\cdots,n\}$ *and* $\{\beta_i:i=1,\cdots,n\}$ *are isotopically equivalent.*

PROOF. Theorem 3.6 will follow from Theorem 3.5 when we show that loop $\alpha_i\cup\beta_i$ is nullhomotopic for $i=1,\cdots,n$. Actually one could show that after a homeomorphism the α_i lie in disjoint product disks for $i=1,\cdots,n$, but we do not need to prove this.

There exist non-nullhomotopic simple loops λ_1 and λ_2 properly embedded in F whose intersection is the single point x_1. We may suppose that if $[\lambda_1]^{n_1}=[\lambda_2]^{n_2}$, $n_1=n_2=0$ since $\chi(F)<0$. We let R be an annulus or möbius band embedded in F and containing λ_1 in its interior. We may suppose that R can be deformation retracted to λ_1. Now the loops in bd$(R\times\{0\})$ are freely homotopic to the corresponding loops in bd$(R\times\{1\})$ in $M-\alpha_1$ and in $M-\beta_1$. We suppose R is a möbius band. Then bd$(R)\times\{0\}$ and bd$(R)\times\{1\}$ bound an annulus B in $M-\beta_1$ by the generalized Dehn's lemma in [5]. Theorem 5.1 in [1] implies the existence of a homeomorphism $h:F\times I\to F\times I$ such that $h(B)=$bd$(R)\times I$. We may think of $h(\beta_1)$ as β_1, $h(\alpha_1)$ as α_1, and $h(A)$ as A and proceed assuming that $B=$bd$(R)\times I$. Similarly bd(R) bounds an annulus A in $M-\alpha_1$. By Theorem 2.5 we may assume that $A=$bd$(R)\times I$. Thus the loop $p(\alpha_1\cup\beta_1)\subset R$ represents

$[\lambda_1]^{n_1}$ for some integer n_1. If R was an annulus, we would have reached the same conclusion by a similar argument. Similarly we can conclude that the loop $p(\alpha_1 \cup \beta_1)$ represents an element $[\lambda_2]^{n_2}$ in $\pi_1(F)$. It follows that the loop $\alpha_1 \cup \beta_1$ is nullhomotopic in M. Similarly the loop $\alpha_i \cup \beta_i$ is nullhomotopic in M for $i=1, \cdots, n$. Thus Theorem 3.6 follows.

One cannot help but notice that our conditions for isotopic equivalence of collections of knotted arcs enable us to confine pairs of arcs to product manifolds. It would be interesting to find conditions implying isotopic equivalence of normal families of arcs which did not allow us to do this. One might also hope to obtain results in manifolds other than $F \times I$ and S^3.

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DEPARTMENT OF MATHEMATICS, VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY, BLACKSBURG, VIRGINIA 24061