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A CHARACTERIZATION OF THE JACOBSON RADICAL IN TERNARY ALGEBRAS

HYO CHUL MYUNG

ABSTRACT. The Jacobson radical Rad T for a ternary algebra T is characterized as one of the following: (i) the set of properly quasi-invertible elements in T; (ii) the set of $x \in T$ such that the principal right ideal $\langle xTT\rangle$ or left ideal $\langle TTx\rangle$ is quasi-regular in T; (iii) the unique maximal quasi-regular ideal in T; (iv) the set of $x \in T$ such that Rad $T^{(x)} = T^{(x)}$. We also obtain ternary algebra-analogs of characterization of the radicals of certain subalgebras in an associative algebra.

1. **Introduction.** Let Φ be a commutative ring with identity. A *ternary algebra* (*τ-algebra*) over Φ is defined as a unital Φ-module T with a trilinear composition $(x, y, z) \to \langle xyz\rangle$ satisfying

$$(1) \qquad \langle\langle xyz\rangle uv\rangle = \langle x\langle yzu\rangle v\rangle = \langle xy\langle zuv\rangle\rangle.$$

In case $\Phi = Z$, τ-algebras are τ-rings defined by Lister [**1**], who extensively studies ring-theoretic structures for τ-rings, in particular, a module theory leading up to the Jacobson radical. Lister defines the Jacobson radical, $R(T)$, for a τ-ring T to be the intersection of kernels of its irreducible modules and shows that $R(T)$ is the intersection of all maximal modular ideals in T [**1**, Theorem 9]. Let T now be a τ-algebra. For $a \in T$, we denote by $T^{(a)}$ the algebra formed by setting

$$x \underset{a}{\cdot} y = \langle xay\rangle$$

on the module T. Then, by (1), $T^{(a)}$ is an associative algebra. An element $x \in T$ is called *properly quasi-invertible* (p.q.i.) in T if it is quasi-invertible (q.i.) in $T^{(a)}$ for all $a \in T$.

DEFINITION. For a τ-algebra T, the Jacobson radical, Rad T, of T is the set of all p.q.i. elements in T.

In this paper we establish τ-algebra-analogs of characterization of the radical in associative or Jordan algebras. Specifically, we will show that Rad T is an ideal of T and coincides with the following: (i) the set of

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$x \in T$ such that $\langle xTT \rangle$ (or $\langle TTx \rangle$) is a quasi-regular right (or left) ideal in T; (ii) $\bigcap_{x \in T}$ Rad $T^{(x)}$; (iii) the set of $x \in T$ such that Rad $T^{(x)} = T^{(x)}$. As applications of this we obtain τ-algebra-analogs of such results as Theorems 4, 6, and 7 given by McCrimmon in [**3**]. Also, the present radical Rad T coincides with $R(T)$ if T is a τ-ring of Lister.

τ-algebras have been called *associative triple systems of* 1*st kind* by Loos [**2**], who has obtained analogous results for *associative triple systems of* 2*nd kind*, defined by the rule

$$\langle\langle xyz\rangle uv\rangle = \langle x\langle uzy\rangle v\rangle = \langle xy\langle zuv\rangle\rangle.$$

The basic example of the 1st kind (or τ-algebra) is a subspace of an associative algebra that is closed relative to $\langle xyz\rangle \equiv xyz$, while the basic example of the 2nd kind is a subspace of an associative algebra with involution, which is closed relative to $\langle xyz\rangle \equiv x\bar{y}z$. Setting $P(x)y = \langle xyx\rangle$, the 1st and 2nd kinds together with the quadratic mapping P become Jordan triple systems (JTS) introduced by K. Meyberg [**4**]. Some of our present results have been proved for JTS in [**4**] where its proofs use complicated identities.

2. **Preliminaries.** Throughout we assume T denotes a τ-algebra over Φ. Any τ-algebra T can be imbedded into an associative algebra $A = T + T^2$ regarded as a τ-algebra [**1**, p. 39]. Then A is called an *enveloping algebra* of T and if $A = T \oplus T^2$, the imbedding is called *direct*. A τ-algebra always possesses a direct enveloping algebra [**1**, p. 38]. If $A = T + T^2$ and $a \in A$, we also denote by $A^{(a)}$ the *a-homotope* of A. Thus if $a \in T$, $T^{(a)}$ is a subalgebra of $A^{(a)}$. Recall $x \in A$ is q.i. in A if $x + y = xy = yx$ for some $y \in A$.

For $x, y \in T$ we introduce the useful operator

$$B(x, y) \equiv \mathrm{Id} - L(x, y) : z \mapsto \langle xyz\rangle$$

for $z \in T$. Then we directly see from (1) the following:

$$B(\langle xyz\rangle, u) = B(x, \langle yzu\rangle), \tag{2}$$

$$B(x, y)B(x, -y) = B(x, \langle yxy\rangle). \tag{3}$$

If $x, y \in T$ and y is a quasi-inverse of x in $T^{(u)}$, then $x + y = \langle xuy\rangle$ and for all $z \in T$,

$$\begin{aligned} B(x, u)B(-y, z)v &= v + \langle yzv\rangle - \langle xuv\rangle - \langle xu\langle yzv\rangle\rangle \\ &= v + \langle yzv\rangle - \langle xuv\rangle - \langle (x + y)zv\rangle \quad \text{(by (1))} \\ &= B(x, u + z)v. \end{aligned}$$

Hence we have the Addition Formula: if $x, y \in T$ are quasi-inverses in $T^{(u)}$, $B(x, u)B(-y, z) = B(x, u + z)$ for all $z \in T$.

LEMMA 1. *If $A=T\oplus T^2$, then, for a, $x\in T$, the following are equivalent*:
(i) *x is q.i. in $T^{(a)}$*;
(i′) *a is q.i. in $T^{(x)}$*;
(ii) *x is q.i. in $A^{(a)}$*;
(ii′) *a is q.i. in $A^{(x)}$*;
(iii) *xa is q.i. in A*;
(iii′) *ax is q.i. in A*;
(iv) *$B(x, a)$ is bijective on T*;
(iv′) *$B(a, x)$ is bijective on T.*

PROOF. That (i)⇒(ii) is obvious, and that (ii)⇒(iii) is a result of McCrimmon [3, Proposition 1]. (iii)⇒(iv): since xa is q.i. in A, the left multiplication $L(1-xa)$ in A is invertible on A (if A does not contain 1, adjoin an identity to A). Then T and T^2 are invariant under $L(1-xa)$, and $B(x, a)$ is the restriction of $L(1-xa)$ to T. Noting that an element in T^2 is q.i. in T^2 if and only if it is q.i. in A since the imbedding is direct, we have that $B(x, a)$ is still invertible on T. (iv)⇒(i): by surjectivity $B(x, a)y=-x$ for some $y\in T$, i.e., $x+y=\langle xay\rangle$. But

$$\begin{aligned} B(x, a)\langle yax\rangle &= \langle yax\rangle - \langle xa\langle yax\rangle\rangle = \langle yax\rangle - \langle\langle xay\rangle ax\rangle \\ &= \langle yax\rangle - \langle xax\rangle - \langle yax\rangle = -\langle xax\rangle \\ &= x + y - \langle xa(x+y)\rangle = B(x, a)(x+y) \end{aligned}$$

and so $x+y=\langle yax\rangle$ too since $B(x, a)$ is injective. By symmetry we prove that (i′)⇔(ii′)⇔(iii′)⇔(iv′). But that (iii)⇔(iii′) is well known for associative algebras, and the proof is complete.

By symmetry in Lemma 1 in x and a, we have a Symmetry Principle for T:

$$x \text{ is q.i. in } T^{(y)} \text{ if and only if } y \text{ is q.i. in } T^{(x)}.$$

From this we get a Shifting Principle for T:

$$x \text{ is q.i. in } T^{(\langle yzu\rangle)} \text{ if and only if } u \text{ is q.i. in } T^{(\langle xyz\rangle)},$$

since, by Lemma 1, x is q.i. in $T^{(\langle yzu\rangle)}$ if and only if $B(x, \langle yzu\rangle)= B(\langle xyz\rangle,u)$ (by (2)) is bijective if and only if u is q.i. in $T^{(\langle xyz\rangle)}$. Hence we obtain the Shifting Theorem for T:

THEOREM 1. *The following conditions for x, y, z, u in a τ-algebra T are equivalent*:
(i) *x is q.i. in $T^{(\langle yzu\rangle)}$*;
(ii) *u is q.i. in $T^{(\langle xyz\rangle)}$*;
(iii) *y is q.i. in $T^{(\langle zux\rangle)}$*;
(iv) *z is q.i. in $T^{(\langle uxy\rangle)}$.*

In view of the equivalence of (i), (ii), and (iii′) in Lemma 1 we can state

LEMMA 2. *If* $A=T\oplus T^2$, *then* $x\in T$ *is p.q.i. in* T *if and only if* xa *and* ax *are q.i. in* T^2 *(or in* A*) for all* $a\in T$.

3. **Characterization of the radical.** A subspace V of T is called a *right* ideal of T in case $\langle VTT\rangle\subseteq V$, a *left* ideal in case $\langle TTV\rangle\subseteq V$, a *medial* ideal in case $\langle TVT\rangle\subseteq V$, and V is called an *ideal* in T if it is right, left, and medial. Thus a right (left) ideal V in T is a right (left) ideal in all $T^{(x)}$. A right ideal V of T is called (right) *quasi-regular* (q.r.) [**1**, p. 45] if $B(v,x)T=T$ for all $v\in V$ and all $x\in T$, and a q.r. left ideal is similarly defined. Noting that V is (right) q.r. in all $T^{(x)}$, this condition is equivalent to that every $v\in V$ is p.q.i. in T.

THEOREM 2. *For a* τ*-algebra* T, *the radical* Rad T *is an ideal of* T.

PROOF. Let $x\in T$ be p.q.i. in T. Then αx is p.q.i. for $\alpha\in\Phi$ since all $B(\alpha x,y)=B(x,\alpha y)$ are bijective. Now, $\langle xyz\rangle$ are p.q.i. for all $y,z\in T$ since $B(\langle xyz\rangle,u)=B(x,\langle yzu\rangle)$, and all $\langle yxz\rangle$ are p.q.i. since all $B(y,\langle xzu\rangle)$ are bijective (Symmetry) and so are all $B(u,\langle yxz\rangle)$ (Shifting). From this, using Symmetry and Shifting Principles, we also see that all $\langle yzx\rangle$ are p.q.i. in T. Finally, let z, u be p.q.i. in T and let $x\in T$. By Symmetry x is q.i. in $T^{(u)}$, and if y is the quasi-inverse of x in $T^{(u)}$ then by the Addition Formula $B(x,u)B(-y,z)=B(x,u+z)$. Since $B(x,u)$ and $B(-y,z)$ are bijective so is $B(x,u+z)$ for all $x\in T$. Thus the set of all p.q.i. elements in T forms an ideal of T.

LEMMA 3. *If* $A=T\oplus T^2$, *the following are equivalent*:

(i) $x\in T$ *is p.q.i. in* T;
(ii) xT *is a q.r. right ideal in* T^2;
(ii′) Tx *is a q.r. left ideal in* T^2;
(iii) *the principal right ideal* $\langle xTT\rangle$ *is q.r. in* T;
(iii′) *the principal left ideal* $\langle TTx\rangle$ *is q.r. in* T.

PROOF. (i)$\Leftrightarrow$(ii). This follows from Lemma 2. (i)$\Leftrightarrow$(iii). If x is p.q.i. in T, all $B(x,\langle yzu\rangle)=B(\langle xyz\rangle,u)$ are bijective and so by Theorem 2 all elements in $\langle xTT\rangle$ are p.q.i., i.e., $\langle xTT\rangle$ is q.r. in T. Conversely, if $\langle xTT\rangle$ is q.r. in T, all $B(\langle xyz\rangle,u)=B(x,\langle yzu\rangle)$ are bijective. Then by (3) all $B(x,t)B(x,-t)$ and $B(x,-t)B(x,t)$ are bijective, so all $B(x,t)$ are too. Since (i) is left-right symmetric, we get (i)$\Leftrightarrow$(ii′)$\Leftrightarrow$(iii′), and the proof is complete.

In view of Theorem 2 and Lemma 3(i), (iii), (iii′), we have the following analogous result of associative algebras (also see [**1**]):

COROLLARY 1. Rad T *is a q.r. ideal in* T *and contains all q.r. right ideals and q.r. left ideals in* T. *Hence* Rad T *is the unique maximal q.r. ideal in* T.

REMARK. In case T is a τ-ring, this corollary has been proved for the radical $R(T)$ at characteristic $\neq 2$ by Lister [**1**, Theorem 10]. Hence Rad T coincides with the radical $R(T)$ defined by Lister.

COROLLARY 2. *If $A=T\oplus T^2$, then*

$$\text{Rad } T = \{x \in T \mid xT \subseteq \text{Rad } T^2\}.$$

PROOF. This follows from Lemma 3 and the well-known fact for associative algebras.

THEOREM 3. *For a τ-algebra T, we have*

$$\begin{aligned}\text{Rad } T &= \{x \in T \mid \langle xTT\rangle \text{ or } \langle TTx\rangle \text{ is q.r. in } T\} \\ &= \{x \in T \mid \text{Rad } T^{(x)} = T^{(x)}\} \\ &= \bigcap_{a\in T} \text{Rad } T^{(a)}.\end{aligned}$$

PROOF. Rad T equals the first set by Lemma 3(i), (iii), (iii′). If $x \in$ Rad T, x is q.i. in all $T^{(\langle zxz\rangle)}$ and by Symmetry all $\langle zxz\rangle$ are q.i. in $T^{(x)}$; so are all z in $T^{(x)}$ since $\langle zxz\rangle = z^2$ in $T^{(x)}$. Hence Rad $T^{(x)}=T^{(x)}$. Conversely, if Rad $T^{(x)}=T^{(x)}$, all y are q.i. in $T^{(x)}$ and by Symmetry x is p.q.i. in T, i.e., $x \in$ Rad T. Thus Rad T equals the second set. If $x \in \bigcap$ Rad $T^{(a)}$, x is q.i. in all $T^{(a)}$ and is p.q.i. in T, while if x is p.q.i. in T, so is x in all $T^{(a)}$ since the y-homotope of $T^{(a)}$ is $T^{(\langle aya\rangle)}$. Hence Rad $T=\bigcap$ Rad $T^{(a)}$ This completes the proof.

4. **Applications.** In this section we use the previous results to characterize the radicals of certain τ-subalgebras of T. A left (or right, or left-right) ideal V of T is also a left (or right, or two-sided) ideal in all $T^{(x)}$, and $V^{(a)}$ for every $a \in T$ is a left, right, or two-sided ideal of $T^{(a)}$, according as V is left, right, or left-right. Hence, in any case, V is a strict inner ideal in all $T^{(x)}$ and $V^{(a)}$ is strictly inner in $T^{(a)}$. We now recall that if K is a strict inner ideal of an alternative algebra B and $x \in K$ is q.i. in B, then x is q.i. in K [**3**, p. 571]. If U is an ideal in a τ-ring T, Lister [**1**, p. 46] proves that $R(U)=U \cap R(T)$. We extend this to left-medial (or medial-right) ideals in a τ-algebra T and give another proof of this as well.

THEOREM 4. *If V is a left-medial (or medial-right) ideal in T, then* Rad $V=V \cap$Rad T.

PROOF. We proceed as in [**3**, Theorem 3]. If $x \in V \cap$Rad T, x is q.i. in all $T^{(a)}$ and so q.i. in all $V^{(a)}$ since $V^{(a)}$ is a left ideal in $T^{(a)}$. Hence x is p.q.i. in V and so $x \in$ Rad V. Conversely, if $x \in$ Rad V, x is q.i. in $V^{(\langle axa\rangle)}$ (so in $T^{(\langle axa\rangle)}$) for all $a \in T$ since V is medial. Hence, by Symmetry, each $\langle axa\rangle$, which is the square of a in $T^{(x)}$, is q.i. in $T^{(x)}$. Thus all a are q.i. in $T^{(x)}$ and by Symmetry $x \in$ Rad T.

For left or right ideals of T, we obtain a result similar to [**3**, Theorem 4]:

THEOREM 5. *If V is a left ideal in T, then*

$$\begin{aligned}\text{Rad } V &= \{z \in V \mid \langle VVz\rangle \subseteq \text{Rad } T\}\\ &= \{z \in V \mid \langle TVz\rangle \subseteq \text{Rad } T\}.\end{aligned}$$

PROOF. We only prove the second equality and the other case is entirely similar. Let $z \in V$ be such that $\langle TVz\rangle \subseteq \text{Rad } T$. Then every $x \in \langle TVz\rangle$ ($\subseteq V$) is q.i. in all $T^{(a)}$ and since $V^{(a)}$ is a left ideal of $T^{(a)}$, x is q.i. in all $V^{(a)}$, so in particular $\langle VVz\rangle$ is a q.r. left ideal of V. Hence by Lemma 3, $z \in \text{Rad } V$. Conversely, if $z \in \text{Rad } V$, let a be any element of V. Then z is q.i. in $V^{(\langle xya\rangle)}$ (so in $T^{(\langle xya\rangle)}$) for all $x, y \in T$, and so, shifting, all x are q.i. in $T^{(\langle yaz\rangle)}$. Hence by Symmetry $\langle yaz\rangle$ is q.i. in all $T^{(x)}$ and $\langle yaz\rangle \in \text{Rad } T$. Thus $\langle TVz\rangle \subseteq \text{Rad } T$.

As usual, a subspace V of T is called an *inner ideal* of T if $\langle aTa\rangle \subseteq V$ for all $a \in V$. Thus all left, right ideals and the subspaces $\langle xTx\rangle$ are inner ideals in T. Furthermore, if V is an inner ideal in T then $V^{(a)}$ is a strict inner ideal in $T^{(a)}$ for all $a \in T$ since $\langle vav\rangle = v^2$ in $V^{(a)}$ for $v \in V$. The following characterization of the radical is applied for all these examples.

THEOREM 6. *If V is an inner ideal in T, then*

$$\text{Rad } V = \{z \in V \mid \langle aza\rangle \in \text{Rad } T \text{ for all } a \in V\}.$$

PROOF. If $z \in V$ and $\langle aza\rangle \in \text{Rad } T$ for all $a \in V$, $\langle aza\rangle \in V$ is q.i. in all $T^{(x)}$ and so is q.i. in all $V^{(x)}$ since $V^{(x)}$ is strictly inner in $T^{(x)}$. In particular, all $\langle aza\rangle$ are q.i. in $V^{(z)}$, but $\langle aza\rangle = a^2$ in $V^{(z)}$ and so all a are q.i. in $V^{(z)}$ (again recall $V^{(z)}$ is strictly inner in $T^{(z)}$). Thus by Symmetry $z \in \text{Rad } V$. The converse is the same as in [**3**, Theorem 7].

An element $e \in T$ is called an *idempotent* if $\langle eee\rangle = e^3 = e$. If $z \in \text{Rad}\langle eTe\rangle$ then $z = \langle exe\rangle$ for some $x \in T$ and so $z = \langle e\langle eze\rangle e\rangle \in \text{Rad } T$ by Theorem 6. Hence we obtain

COROLLARY. *If e is an idempotent in T, then*

$$\text{Rad}\langle eTe\rangle = \langle eTe\rangle \cap \text{Rad } T.$$

5. **Relations with Jordan triple systems.** For a τ-algebra T, let T_J denote the JTS formed from T by setting $P(a)x = \langle axa\rangle$. From $T^{(a)}$, we also form a quadratic Jordan algebra $T^{(a)+}$ by setting $P_a(x) = P(x)P(a)$ and $x^{2(a)} = P(a)x$, and it is well known that $\text{Rad } T^{(a)} = \text{Rad } T^{(a)+}$. Thus, in view of Theorem 3 and [**4**], we have

THEOREM 7. *For a τ-algebra T,* $\text{Rad } T = \text{Rad } T_J$.

As usual, $a \in T$ is called *von Neumann regular* if $a=\langle axa\rangle$ for some $x \in T$, and $b \in T$ is called *trivial* if $\langle bTb\rangle=0$. While one-sided ideals or ideals in T are different from those in T_J, T and T_J have the same inner ideals, idempotents, trivial elements, and von Neumann regularity. Hence, from Theorem 7 and the known result [**4**] for a JTS, we can state

THEOREM 8. *If T is a τ-algebra satisfying the d.c.c. on inner ideals, then the following are equivalent*:

(i) *T is semisimple*;
(ii) *T is von Neumann regular*;
(iii) *T contains no nonzero trivial elements.*

A proof of this for a JTS uses complicated identities [**4**], but some parts of the proof for T are considerably shorter. For example, to show that if $a \in T$ is trivial then $a \in \operatorname{Rad} T$, we simply observe that every element of $\langle aTT\rangle$ is trivial and so $\langle aTT\rangle$ is q.r. in T. Next, to see that if T is von Neumann regular then it is semisimple, let $z \in \operatorname{Rad} T$. Then $z=\langle zaz\rangle$ for some a, so $B(z, a)z=0$; but since $B(z, a)$ is invertible, $z=0$.

Finally, let B be an associative algebra. Then B becomes a τ-algebra on iteration of its product, which we denote by B_τ. Let B^+ and B_J, respectively, denote the quadratic Jordan algebra and the JTS formed from B in the usual manner. Then from Theorems 3 and 7 we obtain

THEOREM 9. *If B is an associative algebra, then*

$$\operatorname{Rad} B = \operatorname{Rad} B^+ = \operatorname{Rad} B_\tau = \operatorname{Rad} B_J.$$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTHERN IOWA, CEDAR FALLS, IOWA 50613