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CERTAIN ELEMENTS IN QUOTIENTS OF MEASURE ALGEBRAS

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ABSTRACT. Let G be a locally compact group, and $M(G)$ the convolution algebra with unit δ of all bounded Radon measures on G. Let I be a left ideal in $M(G)$ and let $C\cap I^{\perp}$ be the space of all bounded continuous functions P on G with $\int_G P\,d\mu=0$ for all μ in I. Suppose that there exists a function P_0 in $C\cap I^{\perp}$ such that $\|P_0\|_\infty > \limsup_{x\to\infty}|P_0(x)|$. Let also λ and μ be two measures in $M(G)$ such that $\|\lambda+I\|=\|\mu+I\|=1$, and $(\lambda+I)*\mu\subset\delta+1$. In this paper we prove under these conditions that there exist a complex number c of modulus one and a point x_0 in G such that $\int_G P\,d\mu=cP(x_0)$ for all functions P in $C\cap I^{\perp}$ with $\|P\|_\infty\leqq|\int_G P\,d\mu|=1$. This generalizes a theorem of K. deLeeuw and Y. Katznelson.

The purpose of this paper is to generalize two results due to K. deLeeuw and Y. Katznelson [1].

Let G be a locally compact (Hausdorff) space, and $C(G)$ the Banach space of bounded continuous functions on G. Let also $C_0(G)$ be the closed linear subspace of $C(G)$ consisting of all continuous functions vanishing at infinity, so that the conjugate space of $C_0(G)$ is $M(G)$, the space of bounded Radon measures on G [2]. Writing

$$\langle P, \mu\rangle = \int_G P\,d\mu \qquad (P\in C(G), \mu\in M(G)),$$

we identify $C(G)$ with a subspace of $M'(G)$, the conjugate space of $M(G)$.

We fix any linear subspace I of $M(G)$, and define

$$\|\mu + I\| = \inf\{\|\nu\| : \nu\in\mu + I\} \qquad (\nu\in M(G)).$$

We also use the following two notations:

$$C\cap I^{\perp} = \{P\in C(G) : \langle P, \lambda\rangle = 0\ (\lambda\in I)\};$$

$$N_\eta(I) = \left\{P\in C\cap I^{\perp} : \|P\|_\infty \leqq 1, \limsup_{x\to\infty}|P(x)| < 1-\eta\right\},$$

where $0<\eta<1$.

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PROPOSITION (cf. [**1**, PROPOSITION 4.1]). *Let μ be any measure in $M(G)$, and suppose that*

$$(*) \qquad \|\mu + I\| - \sup\{|\langle P, \mu\rangle| : P \in N_\eta(I)\} = o(\eta) \quad \text{as } \eta \to 0;$$

then there exists a measure ν in $M(G)$ such that

(i) $\|\nu\| = \|\mu + I\|$, *and*

(ii) $\langle P, \nu\rangle = \langle P, \mu\rangle$

for all functions P in $C \cap I^\perp$.

PROOF. There exists a sequence (ν_n) of measures in $\mu + I$ such that $\lim_n \|\nu_n\| = \|\mu + I\|$. For a given $\varepsilon > 0$, we can take a function P_ε in $N_\eta(I)$ for some $0 < \eta < 1$ such that

$$(1) \qquad \|\mu + I\| - |\langle P_\varepsilon, \mu\rangle| < \varepsilon\eta.$$

Let K_ε be a compact subset of G such that

$$|P_\varepsilon(x)| < 1 - \eta \qquad (x \in K_\varepsilon^c).$$

Then we have

$$\begin{aligned}
|\langle P_\varepsilon, \mu\rangle| &= \lim_n |\langle P_\varepsilon, \nu_n\rangle| \\
&\leqq \liminf_n \left(\int_{K_\varepsilon} + \int_{K_\varepsilon^c}\right) |P_\varepsilon| \, d\,|\nu_n| \\
&\leqq \liminf_n \{|\nu_n|(K_\varepsilon) + (1 - \eta)\,|\nu_n|(K_\varepsilon^c)\} \\
&= \|\mu + I\| - \eta \limsup_n |\nu_n|(K_\varepsilon^c),
\end{aligned}$$

which, combined with (1), yields

$$\limsup_n |\nu_n|(K_\varepsilon^c) \leqq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this assures that there exists a subnet (ν_α) of (ν_n) converging to some measure $\nu \in M(G)$ in the weak topology $\sigma(M(G), C(G))$. In particular, we have

$$\langle P, \mu\rangle = \lim_\alpha \langle P, \nu_\alpha\rangle = \langle P, \nu\rangle$$

for all P in $C \cap I^\perp$, which proves (ii). It remains to show (i). The inequality $\|\nu\| \leqq \|\mu + I\|$ is trivial, and the converse inequality follows from $(*)$ and (ii). This completes the proof.

We now suppose that G is a locally compact group, so that $M(G)$ is a Banach algebra with unit δ under convolution. The following is a generalization of the author's lemma [**4**, Lemma 4.2], which is originally due to K. deLeeuw and Y. Katznelson [**1**, Theorem 1.5] (see also [**3**]).

THEOREM. *Suppose that I is a left ideal in $M(G)$, and that there exists a function P_0 in $C \cap I^{\perp}$ such that*

(a) $$\|P_0\|_\infty > \limsup_{x\to\infty} |P_0(x)| .$$

Let also λ and μ be two measures in $M(G)$ such that

(b) $$\|\lambda + I\| = \|\mu + I\| = 1, \text{ and } (\lambda + I) * \mu \subset \delta + I.$$

Then there exist a complex number c of modulus 1 and a point x_0 in G such that

(i) $$\langle P, \mu\rangle = cP(x_0)$$

for all P in $C \cap I^{\perp}$ with $\|P\|_\infty \leqq |\langle P, \mu\rangle| = 1$.

PROOF. Since I is a left ideal, the space $C \cap I^{\perp}$ is left-translation invariant. Hence, by (a), we may assume that

$$\|P_0\|_\infty = P_0(e) = 1 > \limsup_{x\to\infty} |P_0(x)| ,$$

where e denotes the identity of G. For any given $\varepsilon > 0$, take λ' in $\lambda + I$ so that $\|\lambda'\| < 1 + \varepsilon$. Putting

$$Q_\varepsilon(y) = (1 + \varepsilon)^{-1} \int_G P_0(xy)\, d\lambda'(x) \qquad (y \in G),$$

it is easy to see that Q_ε is in $N_\eta(I)$, where

$$\eta = 1 - \limsup_{x\to\infty} |P_0(x)| .$$

Further, by (b), we have

$$\langle Q_\varepsilon, \mu\rangle = (1 + \varepsilon)^{-1}\langle P_0, \lambda' * \mu\rangle = (1 + \varepsilon)^{-1}\langle P_0, \delta\rangle = (1 + \varepsilon)^{-1}.$$

It follows at once that

$$\sup\{|\langle Q, \mu\rangle| : Q \in N_\eta(I)\} = 1 = \|\mu + I\|.$$

This, combined with Proposition, assures that we can find a measure ν in $M(G)$ such that

(i′) $$\|\nu\| = 1, \quad \text{and} \quad \langle P, \mu\rangle = \langle P, \nu\rangle \qquad (P \in C \cap I^{\perp}).$$

Let us put

$$B = \{P \in C \cap I^{\perp} : \|P\|_\infty \leqq |\langle P, \mu\rangle| = 1\},$$

and fix any function Q in B. It is easy to see from (i′) that

$$|P(x)| = 1 \quad (x \in \operatorname{supp}(\nu), P \in B),$$

and that $P(x)/Q(x) = \text{const}$ on $\operatorname{supp}(\nu)$ for all P in B. Fixing any point x_0 in supp(ν), put $c=\langle Q, \mu\rangle/Q(x_0)$; then we have

$$\langle P, \mu\rangle = \langle P, \nu\rangle = (P(x_0)/Q(x_0))\langle Q, \nu\rangle = cP(x_0).$$

This establishes our theorem.

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