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THE FUNDAMENTAL IDEAL AND π_2 OF HIGHER DIMENSIONAL KNOTS

S. J. LOMONACO, JR.

ABSTRACT. Let $(S^4, k(S^2))$ be a knot formed by spinning a polyhedral arc α about the standard 2-sphere S^2 in the 3-sphere S^3. Then the second homotopy group of $S^4-k(S^2)$ as a $Z\pi_1$-module is isomorphic to each of the following:

(1) The fundamental ideal modulo the left ideal generated by $a-1$, where a is the image in $\pi_1(S^4-k(S^2))$ of a generator of $\pi_1(S^2-\alpha)$.

(2) The first homology group of the kernel of $\pi_1(S^3-k(S^2))\rightarrow \pi_1(S^4-k(S^2))$

0. **Introduction.** A presentation of the second homotopy group of an arbitrary spun knot [3] was calculated as a $Z\pi_1$-module in [4], [5], [6]. Professor John Milnor has conjectured that this is a presentation of the fundamental ideal modulo a suitably chosen element. In this paper, we show that this is actually the case. In particular,

THEOREM 2. *Let $k(S)^2$ be a 2-sphere formed by spinning an arc α about the standard 2-sphere S^2 in the 3-sphere S^3. Let a denote the image in $\pi_1(S^4-k(S^2))$ of a generator of $\pi_1(S^2-\alpha)$ and let $\mathfrak{F}$ be the fundamental ideal in $Z\pi_1$, i.e., the two sided ideal generated by all elements of the form $g-1$ $(g\in\pi_1)$. Then the second homotopy group $\pi_2(S^4-k(S^2))$ as a $Z\pi_1$-module is $\mathfrak{F}/(a-1)$, where $(a-1)$ denotes the left ideal generated by $a-1$. The action of $Z\pi_1$ on π_2 is that induced by left multiplication on $\mathfrak{F}$.*

Moreover, we have yet another characterization of π_2, namely

THEOREM 1. *Let $(S^4, k(S^2))$ be defined as in Theorem 2 above. Then $\pi_2(S^4-k(S^2))$ as a $Z\pi_1$-module is the first homology group of the kernel of $\pi_1(S^3-k(S^2))\rightarrow\pi_1(S^4-k(S^2))$.*

I. **Definition of a spun knot.** Let S^2 be a standard 2-sphere in the 3-sphere S^3 and let α be a polyhedral arc with endpoints lying on S^2 and with interior lying entirely within one of the two components of S^3-S^2.

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Generate a knotted 2-sphere $k(S^2)$ in S^4 by spinning α about S^2 while holding S^2 fixed. (For details see [3], [7].)

II. **Proofs of theorems.** Let $X=S^4-k(S^2)$ and let $X_0=S^3-k(S^2)$ be the 3-dimensional cross-section of the knot. Let X_+ and X_- denote the closures of the two components of $X-X_0$. Similarly, let $X_{00}=S^2-k(S^2)$ and X_{0+} and X_{0-} be the closures of the two components of X_0-X_{00}. Let $\tilde{X}$ be the universal covering of X, and $\tilde{X}_\pm$, $\tilde{X}_0$, $\tilde{X}_{0\pm}$, and $\tilde{X}_{00}$ be the respective lifts of $X_\pm$, X_0, $X_{0\pm}$, X_{00} to $\tilde{X}$.

The lifts $\tilde{X}_\pm$, $\tilde{X}_0$, $\tilde{X}_{0\pm}$ are all connected. This can be seen by inspecting the homotopy sequence for the fibration

$$\pi_1(X) \to \tilde{X}_i \to X_i$$

and noting that $\pi_1(X_i)\to\pi_1(X)$ is onto for $i=+,\ -,\ 0,\ 0+,\ 0-$. Moreover, since $\pi_1(X_i)\to\pi_1(X)$ for $i=+,\ -,\ 0+,\ 0-$ is an isomorphism onto [3], $\tilde{X}_\pm$, $\tilde{X}_{0\pm}$ are all simply connected.

Since $\tilde{X}_\pm$ collapses to $\tilde{X}_{0+}$ via a deformation arising from the spinning, Hurewicz's theorem coupled with the asphericity of knots [2] gives $H_n(\tilde{X}_\pm)=0$ for $n\geqq 1$. Hence, from the Mayer-Vietoris sequence for the triad $(\tilde{X};\ \tilde{X}_+,\ \tilde{X}_-)$, we have

$$H_2(\tilde{X}) \cong H_1(\tilde{X}_0).$$

But by the asphericity of knots [2], $\tilde{X}_0$ is the Eilenberg-Mac Lane space $K(\pi_1(\tilde{X}_0),\ 1)$. Inspecting the homology sequence for the fibration $\pi_1(X)\to \tilde{X}_0\to X_0$, we have

THEOREM 1. *The second homotopy group $\pi_2(S^4-k(S^2))$ as a $Z\pi_1$-module is the first homology group of the kernel of $\pi_1(S^3-k(S^2))\to\pi_1(S^4-k(S^2))$.*

Theorem 2 now follows from a close inspection of the exact sequence

$$0 \to H_1(\tilde{X}_0) \to H_0(\tilde{X}_{00}) \to H_0(\tilde{X}_{0+}) \oplus H_0(\tilde{X}_{0-})$$

arising from the Mayer-Vietoris sequence for $(\tilde{X}_0;\ \tilde{X}_{0+},\ \tilde{X}_{0-})$.

Note that there is a deformation retraction of X_{0+} onto a 2-dimensional CW-complex K_{0+} which induces on X_{00} a deformation retraction onto a 1-dimensional CW-complex K_{00}. The CW-complex K_{00} consists of a single 1-simplex ξ_0 and a single vertex p. The element a of $\pi_1(X)$ carried by ξ_0 is the image of a generator of $\pi_1(X_{00})$. (For details see the lemma of [6].) Thus, $\tilde{X}_{00}$ collapses to

$$\tilde{K}_{00} = \bigcup_{g\in\pi_1} g\xi_0.$$

By direct computation, $Z_0(\tilde{K}_{00})=Z\pi_1 p$ and $B_0(\tilde{K}_{00})=(a-1)p$, where

$(a-1)$ is the left ideal generated by $a-1$. Hence, $H_0(\tilde{X}_{00}) \simeq Z\pi_1/(a-1)$ and

$$0 \to H_1(\tilde{X}_0) \to Z\pi_1/(a-1) \xrightarrow{\sigma_*} J \oplus J.$$

Again by direct computation, $\sigma_* = \varepsilon \oplus (-\varepsilon)$, where $\varepsilon : Z\pi_1/(a-1) \to J$ is the projection of the trivializer [1] (also called the augmentation [**8**]). Thus, $\mathrm{Ker}(\sigma_*) = \mathrm{Ker}(\varepsilon) \cap \mathrm{Ker}(-\varepsilon) = \mathfrak{J}/(a-1)$ and

$$\mathfrak{J}/(a-1) \cong H_1(\tilde{X}_0) \cong H_2(\tilde{X}) \cong \pi_2(X).$$

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State University of New York at Albany, Albany, New York 12203

Current address (S. J. Lomonaco, Jr.): 44 Blue Spruce Lane, Ballston Lake, New York 12019