PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 38, Number 2, April 1973

# ANOSOV DIFFEOMORPHISMS ON NILMANIFOLDS

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ABSTRACT. The purpose of this paper is to give necessary conditions on the map induced by an Anosov diffeomorphism of a nilmanifold on its fundamental group.

We shall generalize a result of Franks [3] which may be rephrased:

THEOREM (FRANKS). *Let $T^n$ be the n-dimensional torus and $f: T^n \to T^n$ an Anosov diffeomorphism. Then $f_*: \pi_1(T^n) \to \pi_1(T^n)$ has no roots of unity as eigenvalues.*

$M$ will always denote a nilmanifold, that is a compact homogeneous space $N/D$ where $N$ is a connected simply connected nilpotent Lie group and $D$ is a uniform discrete subgroup of $N$. Then $\pi_1(M)=D$. Parry describes in [8] how to use the lower central series of $N$ to express $N/D$ as a sequence of torus extensions. We do the same thing with the upper central series $\{e\}=N_0 \subset N_1 \subset \cdots \subset N_{c-1} \subset N_c = N$; in fact there is a sequence of nilmanifolds $N/N_i \cdot D$, $i=0, 1, \cdots, c$, where the dot denotes semi-direct product, and a torus $N_i \cdot D/N_{i-1} \cdot D$ acting on $N/N_{i-1} \cdot D$ by left translation with orbit space $N/N_i \cdot D$. Also $D_i$ in the upper central series of $D$ is just $N_i \cap D$ and $N_i \cdot D/N_{i-1} \cdot D$ is isomorphic to $(N_i/N_{i-1})/(D_i/D_{i-1})$. We recall (Theorem 2 of [7]) that $D_i/D_{i-1}$ is *free* abelian, a fact which is not true in general of the factor groups of the *lower* central series of $D$. Thus $M$ is expressed as a sequence of extensions by tori whose fundamental groups are $D_i/D_{i-1}$.

Let $f$ be a homeomorphism of $M$ and let $f_*$ be the automorphism it induces on the fundamental group $D$. Since we have not mentioned base points yet $f_*$ is only defined up to an inner automorphism of $D$ but that is sufficient for our purposes. $f_*$ preserves the upper central series of $D$ and so induces automorphisms $\varphi_i: D_i/D_{i-1} \to D_i/D_{i-1}$ for $i=1, \cdots, c$. We shall prove

THEOREM. *If $f$ is an Anosov diffeomorphism then none of the $\varphi_i$'s have a root of unity as an eigenvalue.*

Received by the editors August 3, 1972.
*AMS (MOS) subject classifications* (1970). Primary 58F15; Secondary 55F20, 55C20.
*Key words and phrases.* Anosov diffeomorphism, nilmanifold, Lefschetz number, spectral sequence.

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In [**5**] Hirsch proved this for the map induced by $f$ on $H_1(M; R)$. Our proof uses a spectral sequence to calculate the Lefschetz number of $f$ and shows the remarkable fact that it is independent of the twists with which the tori are put together to make up $M$.

I would like to express my thanks to Professor E. C. Zeeman for many helpful conversations and for suggesting the use of spectral sequences, to Professor W. Parry for discussions about nilmanifolds and to the Science Research Council for their financial support.

The theorem for $f$ is equivalent to the theorem for a power of $f$ so (using the fact that $f$ has periodic points, see Proposition 1.7 of [**2**]) we may assume $f$ has a fixed point. Again we may assume this fixed point is the base point $eD$ of $M$ by conjugating $f$ by a translation since $N$ acts transitively on $M$.

By Theorem 5 of [**6**], $f_*: D \to D$ extends uniquely to an automorphism $G: N \to N$. Since $G$ preserves the subgroup $D$ it induces a diffeomorphism $g$ of the homogeneous space $N/D = M$. The diffeomorphisms $f$, $g$ of $M$ induce the same automorphism of the fundamental group $D$ and so, since $M$ is a $K(D, 1)$ (because its universal cover $N$ is contractible, see [**10**, p. 180]), are freely homotopic, see [**12**, Theorem 8.1.11], and induce the same map of $H_*(M)$. Therefore $L(f) = L(g)$.

The automorphism $\varphi_i$ of the fundamental group of the $i$th torus of $M$ is induced by an *automorphism* $g_i$ say of this torus and $g: M \to M$ is the extension of $g_1$ by $g_2$ by $\cdots$ by $g_c$. We show that $L(g) = L(g_1 \times \cdots \times g_c)$. A special case of this was noticed by Bowen [**1**, p. 395]. In fact it follows from the next lemma by induction on $c$ and the observation that the condition about trivial action is satisfied because the series $\{D_i\}$ is *central.*

LEMMA. *Let $\pi: X, * \to B, *$ be a fibre bundle with fibre $F = \pi^{-1} *$ and suppose $\pi_1(B)$ acts trivially on the homology of $F$. Assume that at least one of $B$, $F$ is compact. Let $(\psi, \chi)$ be a bundle map, i.e. a pair of continuous maps s.t. the diagram*

$$\begin{array}{ccc} X, * & \xrightarrow{\psi} & X, * \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ B, * & \xrightarrow[\chi]{} & B, * \end{array}$$

*commutes and let $\omega = \psi | F$. Then $L(\psi) = L(\chi \times \omega)$.*

REMARK. $\psi: X \to X$ and $\chi \times \omega: B \times F \to B \times F$ differ by twists in the fibres so the lemma says that the Lefschetz number ignores these twists. If $\psi = \mathrm{id}_X$ then the result reduces to the multiplicative property of the Euler characteristic (Theorem 9.3.1 of [**12**]) which however is true without the condition of trivial action. This condition is required here since the

Klein bottle $K$ is an $S^1$ bundle over $S^1$ failing to satisfy it and the map $\psi: K \to K$ that induces the identity in the fibre but wraps the base three times round itself has Lefschetz number $-2$ but the corresponding map of $T^2$ has Lefschetz number 0.

PROOF. We use cubical singular homology with real coefficients and the Serre spectral sequence, see [9] and [4]. Let ${}^0_n\square(X)$ be the real vector space with basis all maps of the standard $n$-cube $I^n$ into $X$ such that all vertices are mapped to $*$. Filter ${}^0_n\square(X)$ as follows. Take a basis element $\sigma \in {}^0_n\square(X)$, $\sigma: I^n \to X$ and define $p$ to be the least integer such that $\pi\sigma(u_1, \cdots, u_n)$ is independent of $u_{p+1}, \cdots, u_n$. Then $\sigma \in {}^0_n\square_p(X)$. Now $\psi: X, * \to X, *$ induces a chain map of ${}^0_n\square(X)$ to itself which preserves the filtration by $p$. So $\psi$ induces a map which we denote by $\psi_*$ on every term $E^r_{pq}$ of the spectral sequence obtained from ${}^0_n\square_p$.

Define

$$L(\psi, E^r) = \sum_{p,q} (-1)^{p+q} \operatorname{trace}(\psi_*: E^r_{pq} \to E^r_{pq}).$$

Then $L(\psi, E^r) = L(\psi, E^{r+1})$ by a version of the Hopf trace theorem, see e.g. 5.1.18 of [4].

Now $E^2_{pq} = H_p(B; H_q(F)) = H_p(B) \otimes H_q(F)$ by the assumption of trivial action and $H_n(B \times F) = \bigoplus_{p+q=n} H_p(B) \otimes H_q(F)$ by the Künneth formula. So $L(\psi, E^2) = L(\chi \times \omega)$.

Since one of $B$, $F$ is compact there is an $m$ such that $E^m_{pq} = E^\infty_{pq}$ and

$$\begin{aligned} L(\psi, E^\infty) &= \sum_{p,q} (-1)^{p+q} \operatorname{trace}(\psi_*: E^\infty_{pq} \to E^\infty_{pq}) \\ &= \sum_n (-1)^n \operatorname{trace}(\psi_*: H_n(X) \to H_n(X)) = L(\psi). \end{aligned}$$

Therefore $L(\psi) = L(\psi, E^\infty) = L(\psi, E^m) = L(\psi, E^2) = L(\chi \times \omega)$.

PROOF OF THEOREM. Now we can calculate

$$L(f) = L(g) = L(g_1 \times \cdots \times g_c) = \prod (1 - \lambda)$$

where the product is taken over all eigenvalues $\lambda$ counted with multiplicity of all the maps $\varphi_i$ [11, p. 769]. If one of these eigenvalues is a $j$th root of unity then $L(f^j) = 0$ according to this calculation. But $L(f^j) \neq 0$ if we can show that all the fixed points of $f^j$ have the same Lefschetz index. (Recall that $f$ has a fixed point.) This is easy if the expanding bundle $E^u$ is orientable, see [3, p. 123]. Moreover if $E^u$ is not orientable we can use the same trick as Franks. Namely we construct a covering $f'$ of $f$ on the covering space of $M$ corresponding to that subgroup $H$ of $\pi_1(M)$ which is the inverse image of $2D/[D, D]$ under the Hurewicz map $\pi_1(M) = D \to D/[D, D] = H_1(M; Z)$. Then $f'$ is an Anosov diffeomorphism with orientable expanding bundle so the map induced by $f'$ on $H$ and hence the map induced by $f$ on $D$ has no eigenvalues which are roots of unity. This completes the proof of the theorem.

REMARK. Suppose now $g$ is a *hyperbolic* nilmanifold automorphism. We see from the calculation above that the zeta function $\zeta(g)$ is the same as $\zeta(g_1\times\cdots\times g_c)$ and this is obtained [**11**, p. 769] from the false zeta function $\tilde{\zeta}(t)=\prod(1-\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}t)^{(-1)^{k+1}}$ where the product is taken over all $(i_1,\cdots,i_k)$ s.t. $1\leqq i_1<i_2<\cdots<i_k\leqq n$. Now $g$ induces an automorphism of the Lie algebra of $N$ and if this Lie algebra is not abelian then there must be eigenspaces corresponding to eigenvalues $\lambda_i$, $\lambda_j$ say whose bracket is not zero making $\lambda_i\lambda_j$ an eigenvalue too. So the zeta function above of a toral automorphism can only be the zeta function of a nontoral nilmanifold automorphism if a factor for which $k=1$ cancels with a factor for which $k=2$.

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