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ON KNESER'S ADDITION THEOREM IN GROUPS

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ABSTRACT. The following theorem is proved.

THEOREM A. *Let G be a group written additively with finite nonempty subsets A, B. Assume that B is commutative, i.e. $b_1+b_2=b_2+b_1$, for $b_1, b_2\in B$. Then there exists an Abelian subgroup H of G such that $A+B+H=A+H+B=A+B$ and $|A+B|\geqq|A+H|+|B+H|-|H|$.*

This is Kneser's theorem, if G is Abelian. Also, as an application of the above theorem, the following is proved.

THEOREM B. *Let G be a finite group of order v $(v>1)$ and let $a_1, \cdots, a_v$ be a sequence (repeats are allowed) of nonzero elements of G. The set S of sums $a_{i_1}+\cdots+a_{i_t}$ where $1\leqq i_1<\cdots<i_t\leqq v$ and $1\leqq t\leqq v$ must contain a nontrivial subgroup H of G.*

Finally, the Kemperman d-transform, a transform similar to the Dyson e-transform, is introduced and evidence is given to support the conjecture that Theorem A remains true, if the commutativity of B is dropped.

1. **Introduction.** Let G be an arbitrary group written additively, and let A, B, and C be finite nonempty subsets of G. $A+B$, the Schnirelmann sum of A and B, denotes the set of sums of the form $a+b$ where $a\in A$ and $b\in B$. A set B in G is said to be commutative, if $b_2+b_1=b_1+b_2$ for b_1, $b_2\in B$. Also, $|A|$ denotes the cardinal number of elements in A. Finally, for a set B the notation $\langle B\rangle$ denotes the group generated by B. We follow the notation and terminology in Mann **[3]** and Kemperman **[1]** and **[2]**.

In §2 we prove that Kneser's theorem (cf. Theorem 1.5 in **[3]**) holds in an arbitrary group provided that one of the component sets is commutative. We prove

THEOREM 1. *Let G be an arbitrary group and let A, B denote finite nonempty subsets of G. Assume that B is commutative. Then there exists an Abelian subgroup H of G such that*

$$A+B+H=A+H+B=A+B; \tag{1}$$

$$|A+B|\geqq|A+H|+|B+H|-|H|. \tag{2}$$

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Theorem 1 is Kneser's theorem, if G is Abelian. In §3 we give an application of Theorem 1, and in §4 we discuss the conjecture that Theorem 1 is true, if the commutativity of B is omitted.

We now introduce the Dyson e-transform (cf. pp. 5–6 in [**3**]) and show that its main properties hold, if B is commutative.

DEFINITION 1.1. Let G be a group with finite nonempty subsets A and B. Assume that B is commutative and $0 \in B$. Fix $e \in A$ and define the sets

$$A' = A \cup (e+B), \tag{3}$$

$$B' = B \cap (-e+A). \tag{4}$$

We call A', B' the *Dyson e-transform* of A, B respectively.

LEMMA 1.2. *Let A', B' be the Dyson e-transform of A, B. Then*
(a) $A' \supseteq A$ *and* $B' \subseteq B$,
(b) $A'+B' \subseteq A+B$,
(c) $|A'|+|B'|=|A|+|B|$.

PROOF. Part (a) of the lemma clearly holds. For (c), it suffices to prove that $x \in A' \setminus A$ iff $-e+x \in B \setminus B'$. If $x \in A' \setminus A$, then for some $b \in B$, we have $x=e+b$; hence $-e+x=b$ and $b \notin B'$, otherwise for some $a \in A$ we would have $-e+x=-e+a$, thus $x=a$, a contradiction. So, $x \in A' \setminus A$ implies that $-e+x \in B \setminus B'$. Conversely, if $-e+x \in B \setminus B'$, then for some $b \in B$ we have $-e+x=b$, so $x=e+b$ and $x \notin A$, otherwise for some $a \in A$ we would have $x=a=e+b$, hence $-e+x \in B'$, a contradiction. Thus we have proved $x \in A' \setminus A$ iff $-e+x \in B \setminus B'$, so $|A' \setminus A|=|B \setminus B'|$ which proves (c).

For (b), if $x \in A$, then $x+B' \subseteq A+B$. So we may assume that $x \in A' \setminus A$, hence $x=e+b$ for some $b \in B$. Now $x+B'=e+b+B'$, but b and B' commute because $B' \subseteq B$, so $x+B'=(e+B')+b$. However, $B' \subseteq -e+A$, so $e+B' \subseteq A$. Consequently, $x+B=(e+B')+b \subseteq A+b \subseteq A+B$ which proves (b). This completes the proof of the lemma.

Since the Dyson e-transform holds, if B is commutative, we proceed in a straightforward way (with some minor modifications) following Mann's proof (cf. pp. 6–9 in [**3**]). Further, we introduce some additional notation. For a set A in G, $\bar{A}$ denotes the complement of A in G, i.e. $\bar{A}=G \setminus A$; and for a subgroup H of G, $N(H)$ denotes the normalizer of H in G, i.e. $N(H)=\{x \in G \mid x+H=H+x\}$.

2. **The lemmas and proof of Theorem 1.** Lemmas 2.0, 2.1, and 2.2 below are analogous to Lemmas 1.5.1, 1.5.2, and 1.5.3 respectively in Mann [**3**]. The main point is that the properties of the Dyson e-transform hold, if B is commutative. Hence, by repeatedly applying the e-transform, we

can "essentially" reduce the problem to the case when $A+B=A$. The choice of $H=\langle B\rangle$ then clearly satisfies (1) and (2) in the theorem.

LEMMA 2.0. *Let G be a group with subgroups H_1 and H_2, and let C_1 and C_2 be finite nonempty subsets of G such that*

(a) $C_1+H_1=C_1$,

(b) $C_2+H_2=C_2$,

(c) $N(H_1)\supseteq H_2$ *and* $N(H_2)\supseteq H_1$, *and*

(d) *neither* $\bar{C}_1\cap C_2$ *nor* $\bar{C}_2\cap C_1$ *is empty.*

Then either

$$|\bar{C}_1 \cap C_2| \geqq |H_1| - |H_1 \cap H_2| \quad or \tag{1}$$

$$|\bar{C}_2 \cap C_1| \geqq |H_2| - |H_1 \cap H_2|. \tag{2}$$

PROOF. We first observe that H_1+H_2 is a group with finite normal subgroups H_1, H_2 and $H_1+H_2=H_2+H_1$. This follows from (c) in the lemma. Put $h_1=|H_1|$ and $h_2=|H_2|$. Let $H^*=H_1+H_2$ and $H=H_1\cap H_2$ and put $h=|H|$. Since $H^*/H_2\simeq H_1/H$ and $H^*/H_1\simeq H_2/H$, we have $m_2h=h_1$ and $m_1h=h_2$ where $m_2=|H^*/H_2|$ and $m_1=|H^*/H_1|$. The relations (1) and (2) become

$$|\bar{C}_1 \cap C_2| \geqq h_1 - h = h(m_2 - 1); \tag{3}$$

$$|\bar{C}_2 \cap C_1| \geqq h_2 - h = h(m_1 - 1). \tag{4}$$

Without loss of generality we may assume that $m_2\geqq m_1$. Let $e\in \bar{C}_2\cap C_1$ and form the H^*-coset $R=e+H^*$. R is a union of m_1 disjoint H_1-cosets as well as a union of m_2 H_2-cosets. Conditions (a) and (b) in the lemma imply that C_1 is a union of H_1-cosets and C_2 is a union of H_2-cosets. Assume that $C_1\cap R$ consists of v_1 disjoint H_1-cosets and $C_2\cap R$ consists of v_2 disjoint H_2-cosets. Then $\bar{C}_1\cap R$ consists of m_1-v_1 disjoint H_1-cosets and $\bar{C}_2\cap R$ consists of m_2-v_2 disjoint H_2-cosets. However, one readily verifies using (c) in the lemma that the intersection of an H_1-coset and an H_2-coset in R is exactly an H-coset. Thus $(\bar{C}_1\cap R)\cap(C_2\cap R)$ is a union of $(m_1-v_1)v_2$ disjoint H-cosets and $(\bar{C}_2\cap R)\cap(C_1\cap R)$ is a union of $(m_2-v_2)v_1$ disjoint H-cosets. This yields

$$|\bar{C}_1 \cap C_2 \cap R| = (m_1 - v_1)v_2h; \tag{5}$$

$$|\bar{C}_2 \cap C_1 \cap R| = (m_2 - v_2)v_1h > 0. \tag{6}$$

The inequality in (6) is justified by choice of e.

From (3), (4), (5), and (6) it suffices to prove either

$$(m_1 - v_1)v_2 \geqq m_2 - 1 \quad \text{or} \tag{7}$$

$$(m_2 - v_2)v_1 \geqq m_1 - 1. \tag{8}$$

From (6) we have $v_1 \geqq 1$ and $m_2 \geqq v_2 + 1$.

Now we must have $v_2 \geqq 1$, for if $v_2 = 0$ then (8) holds because $m_2 \geqq m_1$. Also, we must have $m_1 \geqq v_1 + 1$, for if $m_1 = v_1$ then again (8) holds. Consequently from the above we have

$$v_1 \geqq 1, \quad v_2 \geqq 1, \quad m_1 \geqq v_1 + 1, \quad \text{and} \quad m_2 \geqq v_2 + 1. \tag{9}$$

If we deny (7) and (8) and add, we get using (9)

$$\begin{aligned} 2v_1v_2 - 2 > m_1v_2 + m_2v_1 - m_1 - m_2 &= m_1(v_2 - 1) + m_2(v_1 - 1) \\ \geqq (v_1 + 1)(v_2 - 1) + (v_2 + 1)(v_1 - 1) &= 2v_1v_2 - 2, \end{aligned} \tag{10}$$

which is a contradiction. This proves the lemma.

LEMMA 2.1. *Let G be a group with finite nonempty subsets A and B Assume that B is commutative. Let c_0 be a fixed element of $A+B$ where $c_0 = a_0 + b_0$, $a_0 \in A$ and $b_0 \in B$. Then there exist sets A_1, B_1, and an Abelian subgroup H_1 of G satisfying*

(a) $c_0 \in A_1 + B_1 \subseteq A + B$,

(b) $A_1 \supseteq A$ *and* $B_1 \subseteq B$ *(hence B_1 is commutative)*,

(c) $|A_1| + |B_1| = |A| + |B|$,

(d) $H_1 + B_1 = B_1 + H_1$ *and H_1 is a subgroup of* $\langle B \rangle$,

(e) $A_1 + B_1 + H_1 = A_1 + H_1 + B_1 = A_1 + B_1$, *and*

(f) $|A_1 + B_1| \geqq |A_1| + |B_1| - |H_1|$.

Hence from (c) *and* (f), $|A_1 + B_1| \geqq |A| + |B| - |H_1|$.

PROOF. Without loss of generality, we may assume that $a_0 = b_0 = c_0 = 0$. For let $A' = -a_0 + A$ and let $B' = B - b_0$. Note that B' is commutative because B is. Suppose we are supplied by the conclusion of this lemma (for A', B' in place of A, B respectively) sets A_1', B_1', and an Abelian subgroup H_1' satisfying (a) through (f). Then one easily verifies that the choice $A_1 = a_0 + A_1'$, $B_1 = B_1' + b_0$, and $H_1 = H_1'$ satisfies (a) through (f).

If $A + B = A$, then the choice $A_1 = A$, $B_1 = B$, and $H_1 = \langle B \rangle$ satisfies (a) through (f). Hence, we may assume that $A + B \not\subseteq A$. Consequently, there exists $e \in A$ such that $e + B \not\subseteq A$. Fix such an e and consider the Dyson e-transform. From Lemma 1.2 we have

$$\begin{aligned} &A' \supset A \text{ and } B' \subset B, \quad |A'| + |B'| = |A| + |B|, \\ &0 \in A' + B' \subseteq A + B, \quad |B'| < |B|, \end{aligned} \tag{11}$$

where A', B' is the Dyson e-transform of A, B. Then by induction on $|B|$ we are done. This proves the lemma.

One sees that, for each $c \in A+B$, Lemma 2.1 yields an "inner approximation" to $C=A+B$ satisfying (1) and (2) in the theorem. The next lemma combines both Lemma 2.0 and Lemma 2.1 to piece all of these inner approximations together.

LEMMA 2.2. *Let G be a group with finite nonempty subsets A and B. Assume that B is commutative. For each subset $\{c_1, \cdots, c_k\} \subseteq A+B$, there exists a set C_0 and an Abelian subgroup H_0 satisfying*

(a) $C_0+H_0=C_0$,

(b) $H_0+B=B+H_0$ *and* H_0 *is a subgroup of* $\langle B\rangle$,

(c) $\{c_1, \cdots, c_k\} \subseteq C_0 \subseteq A+B$, *and*

(d) $|C_0| \geqq |A|+|B|-|H_0|$.

PROOF. We use induction on k. For $k=1$ the lemma follows from Lemma 2.1. For $k>1$, let C_1, H_1 satisfy (a) through (d) in place of C_0, H_0 where $\{c_1, \cdots, c_{k-1}\} \subseteq C_1$, and similarly let C_2, H_2 satisfy (a) through (d) where $c_k \in C_2$. If either $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$ we are done. Hence, we may assume that neither $\bar{C}_1 \cap C_2$ nor $\bar{C}_2 \cap C_1$ is empty. Since H_1 and H_2 are Abelian subgroups in $\langle B\rangle$, we may apply Lemma 2.0. Thus, either

$$|\bar{C}_1 \cap C_2| \geqq |H_1| - |H_1 \cap H_2| \quad \text{or} \tag{12}$$

$$|\bar{C}_2 \cap C_1| \geqq |H_2| - |H_1 \cap H_2|. \tag{13}$$

If (12) holds, let $H_0=H_1 \cap H_2$ and $C_0=C_1 \cup C_2$. Then $C_0+H_0=C_0$ and $\{c_1, \cdots, c_k\} \subseteq C_0 \subseteq A+B$; furthermore $C_0=C_1 \cup (\bar{C}_1 \cap C_2)$ so $|C_0|=|C_1|+|\bar{C}_1 \cap C_2|$. But $|C_1| \geqq |A|+|B|-|H_1|$ and from (12) we get $|C_0| \geqq |A|+|B|=|H_0|$. If (13) holds the proof is similar. This completes the proof of the lemma.

To prove the theorem, take $\{c_1, \cdots, c_k\}=A+B$ in Lemma 2.2. Inequality (2) in the theorem follows by considering $A+H$, $B+H$ in place of A, B respectively, since $(A+H)+(H+B)=A+B$.

3. An application of Theorem 1.

Examples of commutative sets B in a group G are subsets of the cyclic group $\langle d\rangle$ where $d \in G$ and $d \neq 0$. In particular sets in the form $\{0, d, \cdots, sd\}$, i.e. arithmetic progressions with first term 0, are commutative sets.

As an easy application of Theorem 1, we now prove

THEOREM 3. *Let G be a finite group of order v $(v>1)$ and let $a_1, \cdots, a_v$ be a sequence (repeats are allowed) of nonzero elements of G. The set S of sums $a_{i_1}+\cdots+a_{i_t}$ where $1 \leqq i_1 < \cdots < i_t \leqq v$ and $1 \leqq t \leqq v$ must include a nontrivial subgroup H of G.*

PROOF. Let $S'=\{0, a_1\}+\cdots+\{0, a_v\}$. Then $S'=S$ unless $0 \notin S$. But $0 \in S$ by the following pyramidal argument. Write

$$
\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
&\cdot \\
&\cdot \\
&\cdot \\
s_v &= a_1 + \cdots + a_v.
\end{aligned} \tag{1}
$$

If all the s_i's are distinct then $S=G$, hence $0 \in S$. And, if two are equal say $s_i=s_j$ $(i<j)$, then $a_{i+1}+\cdots+a_j=0$ and again $0 \in S$. Thus $S'=S$ and to prove the theorem we need only estimate S' by Theorem 1, since each $\{0, a_i\}$ is commutative.

Put $A_i=\{0, a_i\}$ for $1 \leqq i \leqq v$. Theorem 1 implies that for each $2 \leqq j \leqq v$ there is a subgroup H of G such that

$$(A_1 + \cdots + A_j) + H = A_1 + \cdots + A_j. \tag{2}$$

If one such subgroup H were nontrivial, then clearly $S' \supset H$ and we are done. If all such subgroups were trivial, then by induction and Theorem 1 we have

$$
\begin{aligned}
|A_1 + \cdots + A_v| &\geqq |A_1| + \cdots + |A_v| - (v - 1) \\
&\geqq 2v - (v - 1) = v + 1,
\end{aligned} \tag{3}
$$

a contradiction since $A_1+\cdots+A_v \subseteq G$ and this concludes the proof of the theorem.

4. The Kemperman *d*-transform and a conjecture. Kemperman (cf. Theorem 5 in **[1]**) proved the following

THEOREM 4. *Let A and B be finite nonempty subsets of a semigroup G such that $k(A, B) \geqq 2$ where $k(A, B)=|A|+|B|-|A+B|$. Let $c_0=a_0+b_0$ ($a_0 \in A$ and $b_0 \in B$) be an element of $A+B$ possessing an inverse in G. Then G contains a finite group H such that*

$$|A + B| \geqq |A| + |B| - |H|, \tag{1}$$

$$a_0 + H + b_0 \subseteq A + B. \tag{2}$$

This result is proved by means of a transform similar to the Dyson *e*-transform. We now proceed to define this transform and examine its properties.

DEFINITION 4.1. Let A, B be finite nonempty subsets of an arbitrary group G. Put $C=A+B$ and $\lambda=|C|-|A|$, $\mu=|C|-|B|$. Assume that $0\in A\cap B$. Let $D=A\cap B$ and let $k(A,B)=|A|+|B|-|C|$.

(3) *Assume that* $A+d\not\subseteq A$ *for some* $d\in D$ *and fix such a* d.

Define the sets

$$A_0=\{a_0\in A \mid a_0+d\notin A\}, \tag{4}$$

$$B_0=\{b_0\in B \mid d+b_0\notin B\}. \tag{5}$$

Put $p=|A_0|$ and $q=|B_0|$. From (3) we have $d\neq 0$, $0\notin A_0$, $0\notin B_0$, and $p\geqq 1$. Define the sets A', B' by

$$A'=A\cup(A_0+d),\quad B'=B\backslash B_0,\qquad \text{if } p\geqq q; \tag{6}$$

$$A'=A\backslash A_0,\quad B'=B\cup(d+B_0),\qquad q>p. \tag{7}$$

We call A', B' the *Kemperman d-transform* of A, B respectively.

In what follows let λ', μ' be defined as λ, μ with A', B' in place of A, B respectively. Also, let $C'=A'+B'$.

LEMMA 4.2. *Let A', B' be the Kemperman d-transform of A, B. Then*
(a) *$A'+B'\subseteq A+B$ and $0\in A'\cap B'\subseteq A'+B'$,*
(b) *$|A'|+|B'|\geqq|A|+|B|$ with equality iff $p=q$,*
(c) *$\lambda'+\mu'\leqq\lambda+\mu$ and either $\lambda'<\lambda$ or $\mu'<\mu$, and*
(d) *$k(A',B')\geqq k(A,B)$.*

PROOF. For (a), we clearly have $0\in A'\cap B'\subseteq A'+B'$. To show $A'+B'\subseteq A+B$ first assume that $p\geqq q$ and consider $(a_0+d)+b'$ where $a_0+d\in A_0+d$ and $b'\in B'$. Then $(a_0+d)+b'=a_0+(d+b')$, but $d+b'\in B$ by definition of B_0, so $a_0+d+b'\in A+B$. The proof for $q>p$ is similar.

For (b), assume that $p\geqq q$. Then $|A'|=|A|+p$ and $|B'|=|B|-q$; hence $|A'|+|B'|=|A|+|B|+(p-q)\geqq|A|+|B|$. The proof for $q>p$ is similar.

For (c), note that $\lambda+\mu=2|C|-|A|-|B|$ and $\lambda'+\mu'=2|C'|-|A'|-|B'|$, but, by (a), $|C'|\leqq|C|$; hence, by (*b*), $\lambda'+\mu'\leqq+\mu$. If $p\geqq q$, then $|A'|>|A|$ hence, by (a), $|C'|-|A'|<|C|-|A|$ thus $\lambda'<\lambda$. The proof for $q>p$ is similar.

For (d), we have $k(A,B)=|A|+|B|-|C|$ and $k(A',B')=|A'|+|B'|-|C'|$. So, by (a) and (b), $k(A',B')\geqq k(A,B)$.

This completes the proof of the lemma.

Using the Kemperman d-transform, we now prove a lemma equivalent to Theorem 4 in the case that G is a group.

LEMMA 4.3. *Let G be a group with finite nonempty subsets A and B. Let c_0 be a fixed element of $A+B$ where $c_0=a_0+b_0$, $a_0\in A$, and $b_0\in B$.*

Then there exist finite subsets A_1, B_1, *and a finite subgroup* H_1 *of* G *such that*

(a) $A_1+B_1\subseteq A+B$,

(b) $a_0\in A_1$, $b_0\in B_1$, *and* $c_0\in A_1+B_1$,

(c) $|A_1|+|B_1|\geqq|A|+|B|$,

(d) $A_1+H_1+B_1=A_1+B_1$,

(e) *either* $H_1\subseteq\langle A_1\rangle$ *or* $H_1\subseteq\langle B_1\rangle$, *and*

(f) $|A_1+B_1|\geqq|A_1|+|B_1|-|H_1|$.

Hence, from (a), (c), *and* (f):

$$|A+B|\geqq|A|+|B|-|H_1|. \tag{8}$$

PROOF. As in the proof of Lemma 2.1, we may assume that $c_0=a_0=b_0=0$. Thus, $0\in A\cap B$ and $C=A+B\supseteq A\cup B$, therefore

$$|C|\geqq|A|+|B|-|D| \tag{9}$$

where $D=A\cap B$. This implies that $|D|\geqq k(A, B)$. If $k(A, B)\leqq 1$, we are done. So, we may further assume that $|D|\geqq k(A, B)\geqq 2$.

Put $\lambda=|C|-|A|$ and $\mu=|C|-|B|$. Clearly $\lambda\geqq 0$ and $\mu\geqq 0$. If $\lambda=0$, then $A+B=A$. The choice $A_1=A$, $B_1=B$, and $H_1=\langle B\rangle$ clearly satisfies the conditions in the lemma. Similarly, the lemma holds if $\mu=0$. Further, if $A+D\subseteq A$, then the choice $A_1=A$, $B_1=B$, and $H_1=\langle D\rangle$ also satisfies the conditions in the lemma where (f) follows from (9). Consequently, we may assume that $A+d\nsubseteq A$ for some $d\in D$. Fix such a d.

By induction on λ and μ, we can assume the lemma holds when A, B is replaced by A', B' respectively such that for the corresponding integers $\lambda'+\mu'\leqq\lambda+\mu$ and either $\lambda'<\lambda$ or $\mu'<\mu$. By means of the Kemperman d-transform (Lemma 4.2), we now construct sets A', B' (the Kemperman d-transform of A, B) such that

(1) $A'+B'\subseteq A+B$ and $0\in A'\cap B'$,

(2) $|A'|+|B'|\geqq|A|+|B|$,

(3) $k(A', B')\geqq k(A, B)\geqq 2$,

(4) $|D'|\geqq 2$, and

(5) $\lambda'+\mu'\leqq\lambda+\mu$ and either $\lambda'<\lambda$ or $\mu'<\mu$.

Therefore, by induction this concludes the proof.

Note, Lemma 4.3 is the analogue of Lemma 2.1. Also, Lemma 4.3 implies that

$$C=\bigcup_{a\in A, b\in B}(a+H_{a,b}+b) \tag{10}$$

where for each $a\in A$, $b\in B$, $H_{a,b}$ is a subgroup of G satisfying

$$|C|\geqq|A|+|B|-|H_{a,b}|. \tag{11}$$

Now, if we could prove a lemma analogous to Lemma 2.0, then we could find a "uniform" H (as in Lemma 2.2) such that

$$A + H + B = A + B; \tag{12}$$

$$|C| \geqq |A| + |B| - |H|. \tag{13}$$

So, on the basis of the above results and Theorem 1, we make the

CONJECTURE. Let G be an arbitrary group and let A, B be finite nonempty subsets of G. Then G contains a finite group H such that

$$A + H + B = A + B; \tag{14}$$

$$|A + B| \geqq |A + H| + |H + B| - |H|. \tag{15}$$

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