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FACTORIZATION OF OPERATORS ON BANACH SPACE

MARY R. EMBRY

ABSTRACT. In this paper it is shown that if D and E are continuous linear operators on a Banach space X, then the following are equivalent: (i) B is a right factor of A, (ii) B majorizes A and (iii) the range of B^* contains the range of A^*.

In [1] Douglas proves the following:

THEOREM (DOUGLAS). *If A and B are operators*[1] *on a Hilbert space X, then the following are equivalent*:

(i) $A=BC$ *for some operator* C *on* X,
(ii) $\|A^*x\|\leqq k\|B^*x\|$ *for some* $k\geqq 0$ *and all* x in X,
(iii) range $A\subset$ range B.

An obvious question to ask is whether this theorem generalizes to the case in which X is an arbitrary Banach space. (In this case we interpret (ii) as $\|A^*x^*\|\leqq k\|B^*x^*\|$ for all x^* in X^*, the adjoint space of X.) The answer to this question is negative, for in [2] Douglas gives an example (which we include later in this paper) of operators A and B for which (iii) is true but (i) is false.

A second, less obvious, question to ask is whether Douglas' theorem remains valid for adjoints of operators on Banach space. The purpose of this note is to present a proof of the following:

THEOREM 1. *Let D and E be operators on a Banach space X. The following conditions are equivalent*:

(i′) $D=FE$ *for some continuous linear transformation* F:range $E\to X$,[2]
(ii′) $\|Dx\|\leqq k\ \|Ex\|$ *for some* $k\geqq 0$ *and all* x *in* X,
(iii′) range $D^*\subset$ range E^*.

Let us note first that this theorem indeed generalizes Douglas' theorem. To see this, let $A=D^*$ and $B=E^*$, which is possible since every Hilbert

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[1] By an operator we mean a continuous linear transformation.
[2] In general F is defined only on the closure of the range of E.

space operator is an adjoint. Then the second and third statements in the two theorems are identical. Condition (i′) in Theorem 1 becomes (i″) $A^*=FB^*$ for some continuous linear transformation F: range $B^*\to X$. But since X is a Hilbert space F has a continuous linear extension G on X so that $A^*=GB^*$. Thus $A=BG^*$ which retrieves (i) of Douglas' theorem.

It is also worthy of note that Theorem 1 remains valid if $D:X\to Y$ and $E:X\to Z$, where each of X, Y, and Z is a Banach space. In that case statement (i′) becomes $D=FE$, where F: range $E\to Y$. This generalization of Theorem 1 may be proven exactly as Theorem 1 will be proven.

To facilitate the proof of Theorem 1 we first prove a lemma, which is a generalization of a theorem of Mac Nerney [**3**, Theorem 1] and which was later rediscovered by Shmulyan [**4**, Lemma 3].

LEMMA 1. *Let $A:X\to Y$ be a continuous linear mapping of Banach spaces. Then $x^*\in$ range A^* if and only if there exists a real number k such that $|x^*(x)|\leqq k\|Ax\|$ for all x in X.*

PROOF. In one direction we argue as follows: if $x^*=A^*y^*$, then $|x^*(x)|\leqq\|y^*\|\ \|Ax\|$. In the other direction we assume that $|x^*(x)|\leqq k\|Ax\|$ for all x in X and define L: range $A\to$ {complex numbers} by $L(Ax)=x^*(x)$. L is well defined, continuous and linear and thus by the Hahn-Banach Theorem has continuous linear extension y^* on Y. Therefore $y^*(Ax)=L(Ax)=x^*(x)$ for all x in X, or equivalently, $A^*y^*=x^*$, showing that $x^*\in$ range A^*.

PROOF OF THEOREM 1. To see that (i′) and (ii′) are equivalent we argue as follows: if $D=FE$, then $\|Dx\|\leqq\|F\|\ \|Ex\|$ for all x in X and if $\|Dx\|\leqq k\|Ex\|$ for all x in X, we define F by $FEx=Dx$. The majorizing condition forces F to be continuous and linear on the range of E.

We shall now show that (ii′) and (iii′) are equivalent. If $\|Dx\|\leqq k\|Ex\|$ for all x in X, then for each x^* in X^* we have $|(D^*x^*)(x)|\leqq k\|x^*\|\ \|Ex\|$ for all x in X, which by Lemma 1 shows that $D^*x^*\in$ range E^*. Thus (ii′) implies (iii′).

Now assume that (iii′) holds. For each positive integer n define $M_n=\{x^*:x^*\in X^*$ and $|x^*(Dx)|\leqq n\|Ex\|$ for all x in $X\}$. Since (iii′) holds, Lemma 1 implies that $\bigcup_n M_n=X^*$. It is easily checked that each M_n is closed in X^*. Consequently the Baire Category Theorem applies and there exist an integer n, a positive number r and an element x_0^* of M_n such that $x^*\in M_n$ whenever $\|x^*-x_0^*\|\leqq r$. Thus if $\|x^*\|\leqq r$, we have for each x in X,

$$2\,|x^*(Dx)|\leqq|(x^*-x_0^*)(Dx)|+|(x^*+x_0^*)(Dx)|\leqq n\,\|Ex\|+n\,\|Ex\|.$$

From this last inequality we see that $|x^*(Dx)|\leqq n\|Ex\|$ for all x in X

whenever $\|x^*\|\leqq r$. Thus for all x^* and all x we have $|x^*(Dx)|\leqq (n/r)\|Ex\|\,\|x^*\|$ and consequently $\|Dx\|\leqq(n/r)\|Ex\|$ for all x in X, completing the proof.

The author is grateful to Professor Douglas for his permission to include the following counterexample:

Douglas' counterexample. Let X be a Banach space, N a subspace of X, and Y the set of bounded functions on the integers so that $f(n)$ is in X for $n\leqq 0$ and $f(n)$ is in X/N for $n>0$. Y is a Banach space with respect to $\|f\|=\sup\|f(n)\|$. Consider the operators A and B on Y defined by

$$\begin{aligned}(Af)(n) &= f(n) \quad \text{for } n=1 \\ &= 0 \quad \text{for } n\neq 1\end{aligned} \qquad \text{and} \qquad \begin{aligned}(Bf)(n) &= \pi f(0) \quad \text{for } n=1 \\ &= f(n-1) \quad \text{for } n\neq 1\end{aligned}$$

where π is the natural map from X to X/N. Then range $A\subset$ range B. Assume that there exists an operator C on Y such that $A=BC$. Let D_1 be the map from X/N to Y and D_2 the map from Y to X defined by

$$\begin{aligned}(D_1x)(n) &= x \quad \text{for } n=1 \\ &= 0 \quad \text{for } n\neq 1\end{aligned} \qquad \text{and} \qquad D_2f=f(0).$$

Then $E=D_2CD_1$ is a map from X/N to X such that $I-E\pi$ is a bounded projection of X onto N. Thus if we choose N to be a subspace for which no bounded projection exists, then we arrive at a contradiction and see that there exists no operator C on Y for which $A=BC$.

Let us now reconsider briefly the implications involved between conditions (i), (ii) and (iii) of Douglas' theorem when X is an arbitrary Banach space. It should be obvious that (i) implies each of (ii) and (iii). By a category argument similar to that used in the proof of Theorem 1, we could show that condition (iii) implies condition (ii). Thus Douglas' counterexample also gives an example in which (ii) is true but (i) is false. The only remaining question concerning the relation of these three conditions is whether (ii) implies (iii). Theorem 1 gives an affirmative answer to this problem if X is reflexive; however, it is an open question whether (ii) implies (iii) in the general case.

ADDENDUM. After the completion of this paper the author received a communication from Richard Bouldin, University of Georgia, indicating that he has found an example of operators on Banach space for which condition (ii) of Douglas' theorem holds, but condition (iii) does not hold.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH CAROLINA, CHARLOTTE, NORTH CAROLINA 28213