ON MATROIDS ON EDGE SETS OF GRAPHS
WITH CONNECTED SUBGRAPHS AS CIRCUITS

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Abstract. It is proved that if $\mathcal{F}$ is a finite family of connected, finite graphs, then a graph $G$ exists such that the subgraphs of $G$ isomorphic to a member of the family cannot be regarded as the circuits of a matroid on the edge set of $G$.

1. In a recent paper [1] we have proved that there are only two matroids on the edge set of any graph $G$ (let us call them edge set matroids), whose circuits are connected subgraphs which form homeomorphic equivalent classes. These matroids are the polygon-matroid, whose circuits are the cycles, and the matroid of bi-circular subgraphs, where a bi-circular graph is a graph formed by two cycles which either have a path in common, or a vertex in common, or are disjoint but linked by a path; these graphs are homeomorphic to those pictured in Figure 1.

\begin{center}
\includegraphics[width=0.5\textwidth]{figure1.png}
\end{center}

Figure 1

The hypothesis concerning homeomorphism is essential to the arguments in [1]. If we drop this hypothesis, the problem of finding all edge set matroids seems to be a very difficult one. As an unknown referee pointed out to me, a matroid of this kind is the matroid whose circuits are: (i) all cycles of even length; (ii) all graphs consisting of two cycles...
of odd length, having only one vertex in common; (iii) all graphs consisting of two cycles of odd length, joined by a path. In any graph $G$, the subgraphs of these kinds are the circuits of a matroid on the edge set of $G$ but a cycle of odd length, although homeomorphic to a cycle of even length, is not a circuit of the matroid.

In this note we prove a theorem concerning edge set matroids. Our terminology is now slightly different from that used in [1]: we reserve the word "circuit" for the matroid-circuits and use "cycle" for simple closed paths in a graph. Moreover a matroid is defined as follows (see Whitney [2]):

Let $E$ be a set of elements and $\mathcal{K}$ a family of subsets of $E$ (circuits). $\mathcal{K}$ defines a matroid on $E$ if and only if the following axioms hold:

**Axiom 1.** No circuit is properly contained in another circuit.

**Axiom 2.** If $K$ and $K'$ are distinct circuits, $a \in K \cap K'$ and $b \in K' - K$, then a circuit $K''$ exists such that $b \in K'' \subset K \cup K' - \{a\}$.

2. Let $\mathcal{F}$ be a family of abstract connected graphs such that in any graph $G$ the subgraphs isomorphic to members of $\mathcal{F}$ are the circuits of a matroid. Call the members of $\mathcal{F}$ circuits. Then

**Lemma 1.** No circuit has a pendant edge.

**Proof.** Let $K$ be a circuit with a pendant edge, say $x$. Take another circuit $K'$, equal to $K$, and let $K \cup K'$ be such that $x$ is the only edge common to $K$ and $K'$ and the pendant vertex of $x$ in each circuit coincides with the vertex of higher degree in the other circuit. Clearly, $x$ is a bridge in $K \cup K'$. By Axiom 2, $K \cup K' - \{x\}$ contains a circuit. But since all circuits must be connected, the existence of such a circuit contradicts Axiom 1.

Thus the lemma is proved.

**Theorem 1.** Let $\mathcal{F}$ be a finite family of connected, finite graphs. Then a graph $G$ exists such that the subgraphs of $G$ isomorphic to a member of $\mathcal{F}$ (or, for brevity's sake belonging to $\mathcal{F}$) cannot be regarded as the circuits of a matroid on the edge set of $G$.

**Proof.** Let $\mathcal{F}$ be a finite family of finite, connected graphs. The members of this family may eventually be regarded as the circuits of a matroid on the edge set of some graphs. However a graph $G$ always exists with a subgraph which, according to the definition of a matroid, must also be a circuit but which does not belong to the family. This is a consequence from the fact that, for the members of a family $\mathcal{F}$ of connected, finite graphs to be circuits of an edge-set matroid defined on any graph $G$, there must always exist a member of $\mathcal{F}$ with a pair of edges of minimal distance arbitrarily large.
To prove it let $K$ be a circuit, $\alpha = (a_1, a_2)$, $\beta = (b_1, b_2)$ two edges of $K$. Consider the four distances $d(a_i, b_j)$ for $i, j = 1, 2$. Let $r$ be the minimal distance between $\alpha$ and $\beta$, and suppose we choose a pair $\alpha, \beta$ in $K$ for which this distance is maximal among all edge pairs. Moreover, without loss of generality, we may suppose $d(a_1, b_3) = r$. There are 6 distinct cases which are summarized in Table I (columns 1 to 5).

<table>
<thead>
<tr>
<th>Cases</th>
<th>$d(a_1, b_1)$</th>
<th>$d(a_1, b_2)$</th>
<th>$d(a_2, b_1)$</th>
<th>$d(a_2, b_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$r (&gt;0)$</td>
<td>$r$</td>
<td>$r$</td>
<td>$r$</td>
</tr>
<tr>
<td>II</td>
<td>$r (&gt;0)$</td>
<td>$r + 1$</td>
<td>$r$</td>
<td>$r$</td>
</tr>
<tr>
<td>III</td>
<td>$r (&gt;0)$</td>
<td>$r + 1$</td>
<td>$r + 1$</td>
<td>$r + 1$</td>
</tr>
<tr>
<td>IV</td>
<td>$r (&gt;0)$</td>
<td>$r + 1$</td>
<td>$r + 1$</td>
<td>$r + 1$</td>
</tr>
<tr>
<td>V</td>
<td>$r (&gt;0)$</td>
<td>$r + 1$</td>
<td>$r + 1$</td>
<td>$r + 2$</td>
</tr>
<tr>
<td>VI</td>
<td>$r (&gt;0)$</td>
<td>$r + 1$</td>
<td>$r + 1$</td>
<td>$r + 1$</td>
</tr>
</tbody>
</table>

Table I

Take another circuit $K'$ and let $K \cup K'$ be such that $\beta$ is the only edge common to $K$ and $K'$. For simplicity let us say the edges of $K$ are black and those of $K'$ are blue. Now, by Axiom 2, a circuit $K''$ exists such that $\alpha \in K'' \subseteq K \cup K' - \{\beta\}$. By Axiom 1, $K''$ contains both black and blue edges. Since $K''$ must be connected and as a consequence of Lemma 1, either $K''$ contains at least one blue path $P(b_1, b_2)$ with length $s \geq 2$, or at least one of the vertices $b_1$ and $b_2$ is a cut-point of $K''$ and there exists at least one cycle in the blue block of $K''$ relative to this cut-point.

If blue paths exist, then take one with minimum length $s \geq 2$. We distinguish two possibilities:

(a) $s \geq 3$. Let $\beta'$ be an edge of $P(b_1, b_2)$ incident to neither $b_1$ nor $b_2$. The minimal distance between $\alpha$ and $\beta'$, which both belong to $K''$, is $\geq r + 1$, that is to say, we obtain a new circuit $K'''$ from a given circuit $K$ with a pair of edges $\alpha$ and $\beta'$ whose minimal distance is greater than the minimal distance between the edges $\alpha$ and $\beta$ of $K$.

(b) $s = 2$. Let $(b_1, x), (x, b_2)$ be the edges in $P(b_1, b_2)$. We have to examine the 6 cases of Table I. In cases I, II, IV and V, we set $\beta' = (b_1, x)$ and $x$ plays now the role of $b_2$. In cases III and VI, we set $\beta' = (x, b_2)$ and $x$ plays the role of $b_1$. The new distances between the endpoints of $\alpha$ and $\beta'$ are given in the columns 6 to 9 of Table I. With this operation we obtain, in cases III and VI, a pair of edges in $K''$, namely $\alpha$ and $\beta'$, whose minimal distance is larger than the distance between $\alpha$ and $\beta$. In the remaining cases, to obtain a circuit with a pair of edges satisfying this condition, one or two iterations of this operation may be required, each time with $K''$ and $\beta'$ in the roles of $K$ and $\beta$, respectively. In fact, cases I and II yield case III, case V yields case VI and case IV yields in a first iteration case V.
which in turn yields case VI. Now from cases III and VI, a new iteration allows us to achieve our aim.

If no blue path exists, then take the above mentioned blue cycle. Suppose the cycle belongs to the blue block of $b_1$. (The same argument holds a fortiori with $b_2$ instead of $b_1$.) Let $\beta'$ be an edge of the cycle non-incident to $b_1$. Obviously, the minimal distance between $\alpha$ and $\beta'$ is $\geq r+1$.

Hence it is always possible to obtain from a pair of edges $\alpha, \beta$ in a circuit $K$, whose distance is $r$, a new pair $\alpha, \beta'$ in a circuit $K''$, whose distance is $\geq r+1$. By repeating the argument, the theorem is proved.

Theorem 1 may also be stated more briefly as follows.

**Theorem 1'**. No edge-set matroid (on an arbitrary graph) may exist with a finite number of connected, finite graphs as circuits.

**References**


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