DISCRETE HAUSDORFF TRANSFORMATIONS

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Abstract. Let $K$ be a complex valued measurable function on $(0, 1]$ such that $\int_0^1 t^{-1/p} |K(t)| \, dt$ is finite for some $p > 1$. Let $H$ be the Hausdorff operator on $l^p$ determined by the moments $\mu_n = \int_0^1 t^n K(t) \, dt$. Define $\Psi(z) = \int_0^1 t^z K(t) \, dt$. Then for each $z$ with $\Re z > -1/p$, $\Psi(z)$ is an eigenvalue of $H^*$. The spectrum of $H$ is the union of $\{0\}$ with the range of $\Psi$ on the half-plane $\Re z \geq -1/p$.

If $\mu$ is a complex Borel measure concentrated on $(0, 1]$ and satisfying $\int_0^1 t^{-1/p} \, d|\mu|(t) < \infty$, then as shown by Hardy [2], $\mu$ determines a bounded operator $H$ on $l^p$, as follows: $(H\mu)(n) = \sum_{k=0}^n \binom{n}{k} \mu_k \, n^{k,a}(k) (n=0, 1, 2, \cdots)$ where $\mu_k = \int_0^1 (1-t)^k \, d\mu(t)$. Furthermore, $\|H\| \leq \int_0^1 t^{-1/p} \, d|\mu|(t)$. In [4], Rhoades studied the operators $H$ which correspond to totally regular Hausdorff summability methods; here the measures $\mu$ are probability measures. He showed that in this case the norm of $H$ on $l^p$ is exactly $\int_0^1 t^{-1/p} \, d\mu(t)$, that $H$ has no eigenvectors in $l^p$, and that $\mu_k = \mu_k 0$ is an eigenvalue of the adjoint $H^*$ (which acts on $l^n$, $1/p + 1/q = 1$). Although the $\mu_k$ depend on $\mu$, the associated eigenvectors do not; indeed, $H^*b_k = \mu_k b_k$ where $b_k = \Delta^k e_0$ and $e_0$ is the unit vector $\{1, 0, 0, 0, \cdots\}$ in $l^n$. Using this observation we are able to extend the results of Rhoades concerning the spectra of the Gamma methods $\Gamma^1_\alpha (a > 1/p)$ to the general class of operators $H$ corresponding to absolutely continuous measures $\mu$. Our theorem expresses the spectrum as the compactified range of the Mellin transform of $t^{1/a} \, d\mu(t)$ over the half-plane $\Im s \geq 0$.

Throughout, $p$ is a real number greater than one.

It is shown in [3] that for each complex number $\lambda$ such that $\Re \lambda^{-1} > 1/q$, the sequence $g_\lambda$ defined by the generating function

$$(1 - w)^{1/\lambda-1} = \sum_{n=0}^\infty g_\lambda(n)w^n$$

belongs to $l^\nu$ (and is an eigenvector of the Cesàro operator). This is

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equivalent to the following: if \( \text{Re } z > -1/p \), then the sequence \( f_z \) given by 
\[
(1-w)^z = \sum_{n=0}^{\infty} f_z(n) w^n
\]
is an element of \( l^p \).

**Theorem 1.** Let \( \Psi(z) = \int_{t=0}^{t^z} \mu(t) \). Then \( \Psi \) is continuous on the half-plane \( \text{Re } z \geq -1/p \) and analytic on \( \text{Re } z > -1/p \). Moreover, for each \( z \) the open half-plane, \( f_z \) is an eigenvector of \( H^* \) corresponding to the eigenvalue \( \Psi(z) \).

**Proof.** We have 
\[
\Psi(z)(1-w)^z = \int_{t=0}^{t^z} (1 - (1-t+wt))^z \mu(t) 
\]
\[
= \int_{t=0}^{t^z} \sum_{n=0}^{\infty} f_z(n)(1-t+wt)^n \mu(t).
\]

Using the binomial theorem and interchanging summation and integration (valid because of uniform convergence), we have 
\[
\Psi(z)(1-w)^z = \sum_{n=0}^{\infty} f_z(n) \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \mu_{k,n-k} w^k.
\]

Since \( H \) has matrix \( \left( h_{rs} \right) \) where \( h_{rs} = \left( \begin{array}{c} s \\ r \end{array} \right) \mu_{s,r} \) if \( s \leq r \), \( h_{rs} = 0 \) if \( r < s \), \( H^* \) has matrix entries \( h^*_{nm} = \left( \begin{array}{c} m \\ n \end{array} \right) \mu_{n,m-n} \) for \( n \leq m \) and 0 otherwise. Hence 
\[
\sum_{n=0}^{\infty} (H^*f_z)(n) w^n = \sum_{n=0}^{\infty} \left( \sum_{m=n}^{\infty} \left( \begin{array}{c} m \\ n \end{array} \right) \mu_{n,m-n} f_z(m) \right) w^n 
\]
\[
= \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} \left( \begin{array}{c} m \\ n \end{array} \right) \mu_{n,m-n} w^n \right) f_z(m).
\]

We conclude that \( H^*f_z = \Psi(z)f_z \).

**Theorem 2.** Suppose that \( d\mu(t) = K(t)dt \) where \( K \) is measurable and 
\[
\int_{0}^{t^{-1/p}} |K(t)| dt < \infty.
\]
Then the spectrum of the Hausdorff transformation \( H \) defined by \( \mu \) on \( l^p \) is precisely the closure of the set of values assumed by 
\[
\Psi(z) = \int_{t=0}^{t^z} t^z d\mu(t)
\]
on the half-plane \( \text{Re } z > -1/p \).

**Proof.** By Theorem 1, the spectrum of \( H \) contains \( \Psi(z) \) for every \( z \) with \( \text{Re } z > -1/p \). Thus it will suffice, in order to prove the theorem, to show that for every \( \lambda \neq 0 \) which is not in the range of \( \Psi \) on \( \text{Re } z \geq -1/p \), the sequence \( 1/(\lambda - \mu_n) \) is the moment sequence of a complex measure \( \nu \) satisfying the integrability condition 
\[
\int_{0}^{t^{-1/p}} d\nu(t) < \infty.
\]

But this is a direct consequence of the Gelfand theory for commutative
Banach algebras. Let us define a function $h$ on the positive axis by setting $h(x)=e^{-x/2}K(e^{-x})$, $0\leq x < \infty$. Then $h \in L^1(0, \infty)$ and by hypothesis, the complex Fourier transform

$$\hat{h}(\xi) = \int_0^\infty e^{i\xi x} h(x) \, dx$$

does not assume the value $\lambda$ anywhere in the half-plane $\Pi=\{\xi: \text{Im } \xi \geq 0\}$. Since the complex homomorphisms of the Banach algebra $L^1(0, \infty)$ are precisely the maps $h \mapsto \hat{h}(\xi)$, $\xi \in \Pi$, it follows from the Gelfand theory that there exists some $h_0 \in L^1(0, \infty)$ such that $\lambda^{-1} h + \lambda h_0 = h \ast h_0$, where $\ast$ represents convolution. (See [1].) In particular, $(\lambda - \hat{h}(\xi))^{-1} = \lambda^{-1} - \hat{h}_0(\xi)$ for all $\xi \in \Pi$. If we set

$$h_0(x) = e^{-x/2}K_0(e^{-x}), \quad 0 \leq x < \infty,$$

then the function $K_0$ so defined satisfies the integrability condition (indeed, $\int_0^\infty t^{-1/p} |K_0(t)| \, dt = \int_0^\infty |h_0(x)| \, dx$), and $(\lambda - \mu)_{-1}$ is the moment sequence of the measure $\nu$, where $\nu = \lambda^{-1} \delta - K_0 \, dt$, $\delta$ the unit mass at 1. Thus the proof is complete.

Note that in view of the analyticity of $\Psi^*$ we can state the conclusion of Theorem 2 as follows: the spectrum of $H$ consists of 0 together with the range of $\int_0^1 t^{1/p} K(t) \, dt$ on the half-plane $\text{Re } z \geq 0$.

In the case of the Gamma methods of order one, $K(t) = at^{a-1}$ where $a>1/p$. The Mellin transform $\Psi(z) = a/(z + a)$, so the spectrum of $\Gamma_a^1$ on $L^p$ is the set of all complex $\lambda$ for which $a/\lambda = (z + 1/p) + a/c$ ($c$=the norm of $\Gamma_a$); i.e., those $\lambda$ such that $\text{Re}(1/\lambda) \geq 1/c = (a - 1/p)/a$. This is the closed disk $|\lambda - c/2| \leq c/2$. So we have Rhoades' description of $\sigma(\Gamma_a^1)$.

**Remarks.** 1. Our result extends to the various interpretations of the limiting case where $p=\infty$. (See [3].)

2. It follows from work of Wallen and Shields (Indiana Univ. Math. J. 20 (1971), 777–788) that each Hausdorff operator on $L^2$ is a bounded analytic function of the Cesàro operator and hence that when $p=2$, Theorem 2 is valid for all $\mu$, absolutely continuous or not. For $p$ in general, it is likely that Theorem 2 is valid for a wider class of measures than those considered here. (The author thanks the referee for these two observations.)

3. James Deddens has independently obtained results which are related to ours.

**References**


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