THE ADDITIVE GROUP OF COMMUTATIVE
RINGS GENERATED BY IDEMPOTENTS1

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Abstract. If \( R \) is a ring, let \( R^+ \) denote its additive group. Our purpose is to give an elementary proof that if \( R \) is a commutative ring generated by idempotents, then any subring of \( R \) generated by idempotents is pure. This yields immediately an independent proof of the following result of G. M. Bergman. If \( R \) is a commutative ring with identity and if \( R \) is generated by idempotents, then \( R^+ \) is a direct sum of cyclic groups.

G. M. Bergman [1, Corollary 4.3] has proved (in particular) that if \( R \) is a commutative ring with identity generated by idempotents, then \( R^+ \) is a direct sum of cyclic groups. This result is rather striking inasmuch as it contains, as a special case when \( R^+ \) is torsion free, the recent celebrated result of G. Nöbeling [2] that the finite-valued functions from any set \( I \) to the integers \( \mathbb{Z} \) form a free abelian group with respect to pointwise addition. Recall that the latter result was first proved for a countable set \( I \) by E. Specker [3] with the aid of the continuum hypothesis.

If \( S \) is a subset of the ring \( R \), we shall use \( \{S\} \) to denote the subring generated by \( S \), while \( \langle S \rangle \) denotes the subgroup of \( R^+ \) that \( S \) generates. An elementary but important fact concerning a commutative ring \( R \) generated by idempotents is the following. If \( A \) is a finitely generated subring of \( R \), then \( A^+ \) is a finitely generated subgroup of \( R^+ \). Since a subgroup of a finitely generated commutative group is again finitely generated and since each element in \( R \) is a linear combination over \( \mathbb{Z} \) of idempotent elements, it suffices to prove the above statement for the case that \( A \) is generated as a ring by a finite number of idempotents. However, in this case, \( A^+ \) is generated by all possible products of the idempotents generation \( A \). Hence \( A^+ \) is finitely generated.

Theorem 1. Let \( R \) be a commutative ring generated by idempotents. Any subring of \( R \) generated by idempotents is pure.

Received by the editors June 6, 1972.


Key words and phrases. Specker theorem, Nöbeling theorem, direct sum of cyclic groups, pure subgroups, commutative rings, idempotent generators.

1 This research was supported, in part, by NSF grant GP-29025.

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**Theorem 2 (Bergman).** If $R$ is a commutative ring generated by idempotents, then $R^+$ is a direct sum of cyclic groups.

**Proof of Theorem 1.** Let $A$ be a subring generated by idempotents. The purity of $A$ in $R$ means that, for each prime $p$ and positive integer $n$, the equation $p^nx = a$ where $a \in A$ has a solution in $A$ whenever the equation has a solution in $R$. Thus it obviously suffices to prove the purity of $A$ in case the subring $A$ is generated (as a ring) by a finite number of idempotents $e_1, e_2, \ldots, e_n$. The purity of $A$ is accomplished by induction on $n$. Equivalently, for an idempotent $e$, we shall prove that $\{A, e\}$ is pure under the assumption that $A$ is a subring generated by $n$ idempotents and that $A$ is pure for any such subring. We can take $A = 0$ to prove inductively the purity of a subring generated by a single nonzero idempotent.

For simplicity of notation, let $B = \{A, Ae\}$ and let $C = \{B, e\} = \{A, e\}$. Since $A(Ae) \subseteq Ae$ and since $Be \subseteq B$, we observe that $B = \langle A, Ae \rangle$ and $C = \langle B, e \rangle$. In order to prove that $C$ is pure, we shall first prove that $B$ is pure. For a prime $p$ and a positive integer $n$, suppose that $p^nx = b$ where $b \in B$ and $x \in R$. Since $B = \langle A, Ae \rangle$, we can write $b = a_1 + a_2e$ where $a_1, a_2 \in A$. Since $p^x(e) = b = (a_1 + a_2)e$ is contained in $Ae$ and since $Ae$ is a subring of $R$ generated by the $n$ idempotents $e_1, e_2, \ldots, e_n$, the induction hypothesis asserts that $p^x(a_3e) = be$ for some $a_3 \in A$. Note that $p^x(x - a_3e) = a_1 - a_1e$. Let $D = \{e_i - e_i e\}$ be the subring of $R$ generated by the idempotents $e_i - e_i e$ where $1 \leq i \leq n$. Since $A = \{e_1, e_2, \ldots, e_n\}$ and since $a_i \in A$, the verification that $a_1 - a_1e$ is contained in $D$ is trivial if we make the observation that $\prod e_i - (\prod e_i) e = \prod (e_i - e_i e)$, where $\prod$ represents a product over any nonempty subset of $\{1, 2, \ldots, n\}$. By the induction hypothesis, we conclude that $p^x d = p^x (x - a_3e)$ for some $d \in D$ since $D$ is generated by $n$ idempotents and since $p^x(x - a_3e) = a_1 - a_1e$ is in $D$. Thus $p^x = p^x (a_3e + d)$, and we have proved the purity of $B$ in $R$ since $a_3e + d \in B$.

Based on the purity of $B$, we can now prove the purity of $C$. Suppose that $p^x = c$ where $c \in C$ and $x \in R$. We want to show that $p^x c_i = c$ for some $c_i \in C$. Write $c = b + p^m e$ where $b \in B$ and $(q, p) = 1$. Since $(q, p) = 1$, $q$ is not relevant to the equation under consideration and therefore there is no loss of generality in assuming that $q = 1$, so now let $c = b + p^m e$. If $m \geq n$, then $p^m b_i = b$ for some $b_i \in B$ due to the purity of $B$. Thus $c_i = b_i + p^m e$ is the desired solution, $p^x c_i = c$. Therefore, we may assume that $m < n$. Hence let $r = n - m > 0$. From the purity of $B$, we can choose $b_1 \in B$ so that $p^m b_i = b$. Suppose, for a positive integer $i$, that we have already shown (as we have for $i = 1$) the existence of $x_i \in \{x\}$ and $b_i \in B$ such that

\[ p^{x + m} x_i = p^m b_i + p^m e. \]
Since $e$ is idempotent, we obtain from (1) the equation
\[ p^r x_i(p^m e) = p^m b_i e + p^m e \]
and finally
\[ p^{r(i+1)+m} x_{i+1} = p^m b_{i+1} + p^m e, \]
where $x_{i+1} = p^{r(i-1)} x_i^2$ and $b_{i+1} = b_i^2 + 2b_i e$. Therefore, (1) has a solution, for all $i \geq 1$, with $x_i \in \{x\}$ and $b_i \in B$. This implies that $p^m e + B^+$ has infinite $p$-height in $(C, x)^+/B^+$. From a remark in the introduction, we know that $(C, x)^+$ is a finitely generated group. Hence $(C, x)^+/B^+$ is a direct sum of cyclic groups since it is finitely generated. We conclude that $p^m e + B^+$ has finite order $t$ where $(t, p) = 1$ since a direct sum of cyclic groups has no elements of infinite $p$-height except elements of finite order relatively prime to $p$. Since $(t, p) = 1$, we may replace the equation $p^r x = b + p^m e$ by the equation $p^r y = b + tp^m e$. However, the latter equation has a solution in $B$ since $tp^m e \in B$ and $B$ is pure. We have shown that $C$ is pure, and this completes the proof of Theorem 1.

The proof of Theorem 2 now follows quickly by induction on the (minimal) cardinality of a generating set of idempotents. Let $E$ be a set of idempotents that generate the ring $R$, and assume that $E$ has been chosen so that $|E|$ is as small as possible. If $E$ is finite, then $R^+$ is finitely generated and, therefore, is a direct sum of cyclic groups. Thus assume that $|F| = \aleph_0$. In view of the purity of any subring of $R$ generated by idempotents and the fact that direct sums of cyclic groups are the pure-projectives in the category of abelian groups, it suffices to prove that $(A, e)^+/A^+$ is a direct sum of cyclic groups whenever $e$ is an idempotent and $A$ is a subring of $R$ generated by idempotents. The point here is that we can ascend to $R$ with a chain
\[ 0 = A_0 \subset A_1 \subset \cdots \subset A_\varepsilon \subset \cdots \]
of subrings such that: (1) $A_{\varepsilon+1}$ is a simple extension of $A_\varepsilon$ by an idempotent, (2) $A_\beta = \bigcup_{\beta < \beta} A_\beta$ if $\beta$ is a limit, and (3) $A_\varepsilon$ is generated by fewer than $\aleph_0$ idempotents. To prove that $(A, e)^+/A^+$ is a direct sum of cyclic groups, we again insert $B = \langle A, Ae \rangle$. Since $B = \langle A, Ae \rangle$, we have the isomorphism $B^+/A^+ \cong \langle Ae/A \cap Ae \rangle^+$ because $A \cap Ae$ is an ideal of $Ae$. Moreover, $Ae/A \cap Ae$ is a commutative ring generated by fewer than $\aleph_0$ idempotents, for by hypothesis the same is true of $A$. By the induction hypothesis, $(Ae/A \cap Ae)^+$ is a direct sum of cyclic groups. Since $B$ is pure in $C = \langle A, e \rangle = \langle B, e \rangle$ and since $C^+/B^+ \cong \langle e \rangle^+/(B \cap \langle e \rangle)^+$ is cyclic, $B^+$ is a direct summand of $C^+$. Therefore, $C^+/A^+$ is a direct sum of cyclic groups since $B^+/A^+$ is. This completes the proof of Theorem 2.
References


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