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ON A GENERALIZED VOLTERRA EQUATION IN HILBERT SPACE[1]

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ABSTRACT. The objective of this paper is to establish a fundamental theorem on the existence and uniqueness of the solution of a generalized Volterra equation in Hilbert space. Some applications to the areas of abstract functional analysis and modern mathematical systems theory are pointed out.

1. **Introduction.** A bounded linear operator P on a Hilbert space H is called an *orthoprojector* if for all pairs $x, y \in H^2$ one has $\langle Px, y\rangle = \langle x, Py\rangle$, and $P^2x = Px$. If P_1 and P_2 are two orthoprojectors, the symbol $P_1 < P_2$ is used to indicate that P_1H (the range of P_1) is contained in P_2H. A set $\mathfrak{P}$ of orthoprojectors is called a *chain* if for every pair $P_1, P_2 \in \mathfrak{P}$ one has either $P_1 < P_2$ or $P_2 < P_1$; the chain $\mathfrak{P}$ is called *bordered* if it contains the null operator 0, and the identity operator I; $\mathfrak{P}$ is *closed* if it has the property that whenever a sequence of orthoprojectors $\{P_i\} \in \mathfrak{P}$ is such that $\lim\{P_i x\} = Px$ for all $x \in H$, then $P \in \mathfrak{P}$. A chain z is a *partition* of $\mathfrak{P}$ if z is composed of a finite number of orthoprojectors in $\mathfrak{P}$. If z_1 and z_2 are two partitions of $\mathfrak{P}$, the symbol $z_2 \supset z_1$ will indicate that if $P \in z_1$, then $P \in z_2$.

Suppose that $\Phi(\cdot)$ is an operator valued function which associates to each partition z of $\mathfrak{P}$ a bounded operator $\Phi(z): H \to H$. The function $\Phi(\cdot)$ is said to have as *limit in the norm* the operator T, if given any $\varepsilon > 0$ it is possible to find a partition z_ε of $\mathfrak{P}$ such that for every other partition z with the property $z \supset z_\varepsilon$, one has $|\Phi(z) - T| < \varepsilon$.

Suppose that Y is a bounded operator in H, where H is equipped with a bordered and closed chain $\mathfrak{P}$, and consider the operator valued function $S(\cdot)$ defined as follows: if $z = \{0 = P_0 < P_1 \cdots < P_{n-1} < P_n = I\}$ is a partition of the bordered and closed chain $\mathfrak{P}$, then $S(z) = \sum_{i=1}^{n} \Delta P_i Y P_{i-1}$, where $\Delta P_i = P_i - P_{i-1}$. If $S(\cdot)$ has as its limit in norm some operator T, then it will be said that the *integral* $(m) \int_{\mathfrak{P}} dP\, YP$ converges and equals T. The integrals $(m) \int_{\mathfrak{P}} PY\, dP$, and $\int_{\mathfrak{P}} dP\, Y\, dP$ are defined analogously.[2]

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[2] All the concepts presented up to this point are taken directly from the book of Gohberg and Kreĭn [5, §§1.3 and 1.4] and are included here to make the paper self-contained.

Our purpose is to state, prove and discuss the following theorem.

THEOREM 1. *Let the Hilbert space H be equipped with the bordered and closed chain of orthoprojectors* $\mathfrak{B}$, *and suppose that X is a linear bounded operator such that* $X=(m)\int_{\mathfrak{B}} dPXP$, *and Y is a bounded and Lipschitz continuous operator such that* $PY=YP$ *and* $(I-P)Y=Y(I-P)$ *for all* $P\in\mathfrak{B}$. *Then the inverse operator* $(I-XY)^{-1}$ *exists, is bounded and continuous and it has the property that* $P(I-XY)^{-1}=P(I-XY)^{-1}P$ *for all* $P\in\mathfrak{B}$.

To help illustrate the statement of the theorem, and the notational machinery which goes with it, it seems appropriate to conclude the section with the following simple example.

EXAMPLE. Let H be given by $L_2(0, +\infty)$, the Hilbert space of real valued Lebesgue square integrable functions defined on the interval $(0, +\infty)$, and suppose that the operators Y and X are defined as follows: if $x, y, w, z\in L_2(0, +\infty)$, and $y=Yx$, $z=Xw$, then:

$-y(t)=N_l(x(t))$, where $N_l(\cdot)$ is any bounded and Lipschitz continuous real valued function defined on the real line;

$-z(t)=h(t)[\sum_{n=0}^{\infty} g_n w(t-\Delta t_n)]$, where $h(\cdot)\in L_\infty(0, \infty)$, and $\lim_{t\to\infty}\text{ess sup}|h(t)|=0$, $\Delta t_0>0$, $\Delta t_{n-1}<\Delta t_n$, and $\{g_n\}$ is a summable sequence of real numbers ($\sum_{n=0}^{\infty}|g_n|<+\infty$).

In $L_2(0, +\infty)$ we can consider the bordered and closed chain of orthoprojectors, $\mathfrak{B}$, given by the family of truncation operators $\{P^t\}$, $t\in[0, +\infty]$, defined as follows: if $x, y\in L_2(0, +\infty)$ and $x=P^t y$, then $x(s)=y(s)$ for $s\in(0, t)$, and $x(s)=0$ for $s\in(t, +\infty)$; when $t=\infty$ then $P^\infty x=x$. Note that Y is a bounded and Lipschitz continuous (not necessarily linear) operator, and X is linear and bounded (not necessarily compact); moreover, it is not difficult to recognize that Y has the property that $PY=YP$ and $(I-P)Y=Y(I-P)$ for each $P\in\mathfrak{B}$, while X is such that $X=(m)\int_{\mathfrak{B}} dPXP$. Clearly then, all the hypotheses of the theorem are satisfied, and one can conclude that the operator $(I-XY)^{-1}$ is well defined, bounded and continuous, and it enjoys the property that $P^t(I-XY)^{-1}=P^t(I-XY)^{-1}P^t$ for every $t\in[0, +\infty]$.

2. **Applications.** An example of application of Theorem 1 to the area of abstract functional analysis is given by the following corollary.

COROLLARY 1. *Let the Hilbert space H be equipped with the bordered and closed chain of orthoprojectors* $\mathfrak{B}$, *and suppose that X is a linear and bounded operator for which the integral* $Y=(m)\int_{\mathfrak{B}} PX\,dP$ *converges. Then for every scalar* α, *the operator* $(I-\alpha Y)^{-1}$ *exists and it is bounded and continuous.*

PROOF. From $Y=(m)\int_{\mathfrak{B}} dPXP$ it follows $\int_{\mathfrak{B}} dP\,Y\,dP=0$, and $(I-P)Y=(I-P)Y(I-P)$. This leads to $\int_{\mathfrak{B}'} dP'\,Y\,dP'=0$, and $P'Y=P'YP'$ for all $P'\in\mathfrak{B}'$, where $\mathfrak{B}'$ is the chain of orthoprojectors defined as follows: if $P\in\mathfrak{B}$ then $I-P=P'\in\mathfrak{B}'$. These two properties of Y are equivalent to $Y=(m)\int_{\mathfrak{B}} dP'\,YP'$, and the desired result is an immediate consequence of Theorem 1.

Observe that Corollary 1 gives a generalization of previous theorems due to Brodskiĭ and to Gohberg and Kreĭn. In particular this result was first obtained by Gohberg and Kreĭn in the case where X is a Hilbert Schmidt operator. Brodskiĭ [**1**] extended its validity to the case where X is linear and completely continuous and the $\mathfrak{B}$ is maximal.[3] Gohberg and Kreĭn further improved Brodskiĭ's result by lifting the requirement about the maximality of $\mathfrak{B}$ [**5**, Theorem 6.1b, p. 27].

To appreciate some systems theory implications of Theorem 1, note that the study of the inverse operator $(I-K)^{-1}$ is equivalent to the study of the closed loop feedback system illustrated in Figure 1. A linear operator

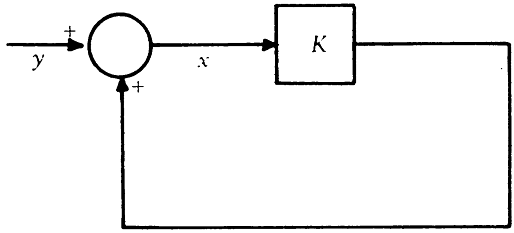

FIGURE 1

X such that $X=(m)\int_{\mathfrak{B}} dPXP$ is referred to as *strictly causal*, and an operator Y satisfying the relation $PY=YP$ and $(I-P)Y=Y(I-P)$ for all $P\in\mathfrak{B}$ is called *memoryless* (see for example [**6**]). According to a popular criterion of stability of a feedback system, see for example [**2**], Theorem 1 can now be rephrased as follows.

COROLLARY 2. *A sufficient condition for the closed loop feedback system in Figure* 1 *to be stable is that K is given by the composition of a memoryless and Lipschitz continuous system with a linear bounded and strictly causal system.*

It is of interest to note that the systems considered in the above corollaries are somewhat similar to those which are considered, for example,

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[3] A chain $\mathfrak{B}$ is *maximal* if there exists no other chain $\mathfrak{B}'$ such that $\mathfrak{B}\subset\mathfrak{B}'$.

in the celebrated Popov Stability Criteria. For a discussion about the relationship between results of this latter type and those of the present development the reader is referred to [**3**] and [**4**].

3. **Some preliminary results.** To prove Theorem 1 it is helpful to recall the following lemmas. In stating these lemmas, the symbols X, Y, and K indicate bounded operators on a Hilbert space H, where H is assumed to be equipped with the bordered and closed chain of orthoprojectors $\mathfrak{P}$.

LEMMA 1. *Necessary and sufficient conditions for X to be such that $X=(m)\int_{\mathfrak{P}} dPXP$, are that $\int_{\mathfrak{P}} dPX\,dP=0$, and $PX=PXP$ for all $P\in\mathfrak{P}$.*

LEMMA 2. *If X is such that $X=(m)\int_{\mathfrak{P}} dPXP$, and Y is such that $PY=PYP$ for all $P\in\mathfrak{P}$, then XY is such that $XY=(m)\int_{\mathfrak{P}} dPXYP$.*

LEMMA 3. *If Y is such that $PY=YP$ and $(I-P)Y=Y(I-P)$ then for every pair $P_1, P_2\in\mathfrak{P}$ one has $(P_2-P_1)Y=(P_2-P_1)Y(P_2-P_1)$.*

LEMMA 4. *If K is Lipschitz continuous and has a Lipschitz norm less than 1, then the inverse $(I-K)^{-1}$ exists and it is bounded and continuous. Moreover, if $PK=PKP$ for all $P\in\mathfrak{P}$, then one also has $P(I-K)^{-1}=P(I-K)^{-1}P$.*

The details of the proofs of Lemmas 1, 2, and 3 can be found in [**3**], and are omitted for brevity. It suffices here to say that Lemmas 1 and 2 follow from the definition of integral, and Lemma 3 is a consequence of the fact that if $PY=YP$ and $(I-P)Y=Y(I-P)$, then $PY=PYP$ and $(I-P)Y=(I-P)Y(I-P)$. In regard to Lemma 4, a sketch of its proof is given in the Appendix.

4. **Outline of the proof.** Let us pose $K=XY$ and let us start to show that the inverse operator $(I-K)^{-1}$ exists, that is: for every $y\in H$ the equation

$$y = x - Kx \tag{1}$$

has a unique solution $x\in H$. Noting that $PY=PYP$, and applying Lemmas 1 and 2 one has

$$PK = PKP, \quad \text{for all } P\in\mathfrak{P}, \tag{2}$$

and

$$\int_{\mathfrak{P}} dPK\,dP = 0. \tag{3}$$

This latter equation implies that there exists a partition z of $\mathfrak{B}$, $z=\{0=P_0, P_1, \cdots, P_N=I\}$, such that

$$|\Delta P(i)K\,\Delta P(i)| < 1, \tag{4}$$

where $i=1, 2, \cdots, N$, and $\Delta P(i)=P_i-P_{i-1}$.

Observe now that, by (2), the problem of finding a solution to (1) is equivalent to finding an $x \in H$ such that the following equations are satisfied

$$\Delta P(i)y = \Delta P(i)x - \Delta P(i)KP_i x \tag{5}$$

where $i=1, 2, \cdots, N$. In the case of $i=1$, (5) becomes

$$\Delta P(1)y = \Delta P(1)x - \Delta P(1)K\,\Delta P(1)x, \tag{6}$$

where, by (4), the operator $\Delta P(1)K\,\Delta P(1)$ is Lipschitz continuous, and has a Lipschitz norm less than 1. From Lemma 4, there exists then a well-defined and bounded operator $T^1:\Delta P(1)H \to \Delta P(1)H$, such that the element

$$\Delta P(1)x = T^1\,\Delta P(1)y \tag{7}$$

is the unique solution to (6) in $\Delta P(1)H$. Similarly, for $i=2$, (5) becomes

$$\Delta P(2)y = \Delta P(2)x - \Delta P(2)KP_2 x. \tag{8}$$

Using the linearity of X and Lemmas 1 and 3, this latter equation can be rewritten as follows

$$\Delta P(2)y + \Delta P(2)K\,\Delta P(1)x = \Delta P(2)x - \Delta P(2)K\,\Delta P(2)x \tag{9}$$

where, by (4), $\Delta P(2)K\,\Delta P(2)$ is Lipschitz continuous with Lipschitz norm less than 1, and $\Delta P(1)x$ is given by (7). Lemma 4 can then be applied again and there exists a bounded continuous operator $T^2:\Delta P(2)H \to \Delta P(2)H$ such that, posing

$$\Delta P(2)x = T^2\,\Delta P(2)[y + K\,\Delta P(1)x], \tag{10}$$

one obtains the unique solution to (8), $P_2x=\Delta P(1)x+\Delta P(2)x$. Proceeding by induction, suppose that, for $j=1, 2, \cdots, i-1$, (5) has the unique solutions $P_jx=P_{j-1}x+\Delta P(j)x$. Then, for $j=i$, (5) has the unique solution $P_ix=P_{i-1}x+\Delta P(i)x$, where

$$\Delta P(i)x = T^i\,\Delta P(i)[y + KP_{i-1}x] \tag{11}$$

and $T^i:\Delta P(i)H \to \Delta P(i)H$ is a well-defined bounded and continuous operator. This recursive relation defines the element $x=\sum_{i=1}^N \Delta P(i)x$, and this element is the unique solution to (1).

To prove that $(I-K)^{-1}$ is bounded, suppose that x, $y \in H$ and $x = (1-K)^{-1}y$. Observe that, for each $i=1, 2, \cdots, N$, $\Delta P(i)x = \Delta P(i)(I-K)^{-1}y$ is defined by (11). Moreover, using the boundedness of K and applying Lemma 4, it follows that there exists a set of positive real numbers M_1, M_2', M_2'', M_3', M_3'', $\cdots$, M_i', M_i'', such that

$$|\Delta P(1)(I - K)^{-1}y| \leqq M_1 |y|$$

and for $i=2, 3, \cdots, N$,

$$|\Delta P(i)(I - K)^{-1}y| \leqq M_i' |y| + M_i'' |P_{i-1}x|.$$

From these equations it follows that there exists a positive M such that

$$|x| = |(I - K)^{-1}y| \leqq M |y|.$$

In regard to the continuity of $(I-K)^{-1}$, this continuity is equivalent to that of the operators $\Delta P(i)(I-K)^{-1}$, $i=1, 2, \cdots, N$. On the other hand, $\Delta P(i)(I-K)^{-1}$ can be computed using (11) and therefore, in view of the continuity of T^i, it is continuous if $P_{i-1}x$ depends continuously on y. But, from (7), $P_1 x = \Delta P(1)x$ does depend continuously on y. Similarly from (10), $P_2 x = \Delta P(2)x + \Delta P(1)x$ depends continuously on y. By induction and (11) it follows that P_{i-1} depends continuously on y for each $i=2, 3, \cdots, N$.

It remains to show that, for each $P \in \mathfrak{B}$, one has

$$P(I - K)^{-1}y = P(I - K)^{-1}Py. \tag{12}$$

To this purpose, consider the partition $z' = z \cup \{0, P, I\} = \{0 = P_0, P_1, \cdots, P_{i-1}, P, P_i, \cdots, P_N = I\}$. Using the notations

$$x^1 = (I - K)^{-1}y^{-1}, \quad x^2 = (I - K)^{-1}y^2,$$

where $y^1 = y$, and $y^2 = Py$, from (11) one has

$$\begin{aligned} x^q = (I - K)^{-1}y^q = &\sum_{j=1, j\neq i}^{N} T_j[\Delta P(j)(y^q + KP_{j-1}x^q)] \\ &+ T_i'[(P - P_{i-1})y^q + KP_{i-1}x^q] \\ &+ T_i''[(P_i - P)y^q + KPx^q], \end{aligned}$$

where $q=1, 2$. By direct inspection, it follows

$$P_1x^1 = P_1x^2, \quad P_2x^1 = P_2x^2, \cdots, P_{i-1}x^1 = P_{i-1}x^2, \quad Px_1 = Px_2.$$

This implies the validity of (11).

5. **Appendix.** In what follows we give a sketch of the proof of Lemma 4. To this purpose, we prove first that the inverse of $(I-K)$ exists. To do this it is clearly sufficient to show that, for every $y \in H$, the equation

$$y = x - Kx \tag{A.1}$$

has one and only one solution $x \in H$. This, in turn, is equivalent to showing the existence and uniqueness of a fixed point for the operator $T_y: H \to H$, defined by the relation $T_y(x)=y+Kx$. But T_y is a contraction operator because for any pair $x_1, x_2 \in H$ one has

$$|T_y(x_1) - T_y(x_2)| = |Kx_1 - Kx_2| \leqq l\,|x_1 - x_2|,$$

where $l<1$ is the Lipschitz norm of K. The desired result follows then by the Banach contraction mapping theorem. In regard to the boundedness and continuity of $(I-K)^{-1}$, note that $(I-K)^{-1}0=0$, and it will be then sufficient to prove that $(I-K)^{-1}$ is Lipschitz continuous. To this purpose, observe that for any pair $x_1, x_2 \in H$ one has

$$\text{(A.2)} \quad \begin{aligned} &|x_1 - Kx_1 - (x_2 - Kx_2)| \\ &\qquad \geqq |x_1 - x_2| - |Kx_2 - Kx_1| \geqq (1 - l)\,|x_1 - x_2| \end{aligned}$$

where $l<1$ is once again the Lipschitz norm of K. Using the fact that $(I-K)$ is invertible, and posing $y_1=(I-K)x_1$, and $y_2=(I-K)x_2$, (A.2) implies that for every pair $y_1, y_2 \in H$ one has

$$|y_1 - y_2| \geqq (1 - l)\,|(I - K)^{-1}y_1 - (I - K)^{-1}y_2|,$$

that is,

$$|(I - K)^{-1}y_1 - (I - K)^{-1}y_2| \leqq (1 - l)^{-1}\,|y_1 - y_2|.$$

To complete the proof of Lemma 4, it remains to be checked that if one has $PK=PKP$ for $P \in \mathfrak{B}$, then one also has $P(I-K)^{-1}=P(I-K)^{-1}P$. This is clearly equivalent to showing that if $y_1, y_2 \in H$ and $Py_1=Py_2$ then

$$\text{(A.3)} \qquad P(I - K)^{-1}y_1 = P(I - K)^{-1}y_2.$$

Posing $x_1=(I-K)^{-1}y_1$, and $x_2=(I-K)^{-1}y_2$, and applying the hypotheses that $PK=PKP$, one has

$$Py_1 = Px_1 - PKPx_1, \quad \text{and } Py_2 = Px_2 - PKPx_2.$$

Since $Py_1=Py_2$, these equalities lead to

$$|Px_1 - Px_2| = |PKPx_2 - PKPx_1|$$

and, by the hypotheses that the Lipschitz norm of K is less than 1, one can conclude that $Px_1=Px_2$, and therefore that (A.3) is true.

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