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THE DUAL AND BIDUAL OF CERTAIN A^*-ALGEBRAS

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ABSTRACT. It is well known that every B^*-algebra is Arens' regular and that its bidual is a B^*-algebra. Wong has asked whether a dual A^*-algebra of the first kind is Arens' regular. It is shown that this is true in the topologically simple case; in the course of the proof it is shown that in this case the bidual is, modulo its radical, an A^*-algebra of the first kind.

1. **Introduction.** Let $(A, \|\ \|, |\ |)$ be an A^*-algebra where $\|\ \|$ is the Banach algebra norm and $|\ |$ is the auxiliary norm, satisfying the B^* condition. Following [**1**] and [**4**] we say that A is of the first kind if it is an ideal of its completion, $\mathscr{A}$, with respect to $|\ |$. Since $\|\ \|$ majorises $|\ |$ it is easy to show (see Lemma 0 below) that A is of the first kind if and only if there is a constant $k_0(A)$ that satisfies

$$\|abc\| \leqq k_0(A)\,|a|\,\|b\|\,|c|, \qquad a, b, c \in A.$$

A topologically simple dual A^*-algebra has a faithful continuous $*$-representation on a Hilbert space, H, with its dense socle corresponding to the set, $F(H)$, of all finite rank operators on H; this representation is an isometry for the auxiliary norm. Accordingly we shall assume in future that A is a selfadjoint algebra of operators on H, that A is a Banach algebra with respect to a norm, α, that majorises the operator norm, λ, and that A is an ideal of $K(H)$, the compact operators on H. Following the proof of Theorem 3.1 in [**1**] we see that A is then an ideal of $B(H)$ and hence that there is a constant $k(A)$ that satisfies

$$\alpha(ABC) \leqq k(A)\lambda(A)\alpha(B)\lambda(C)$$

for all A, C in $B(H)$ and $B \in A$. Standard notations and terminology and the relevant theory of A^*-algebras will be found in [**5**]. Other notations which we shall adopt are $\tau c(H)$ for the trace class of operators on H, τ for the trace norm and t for the trace. Following [**3**] we say that a norm p on $F(H)$ is *modular* if, for some constant, k,

$$p(ABC) \leqq k\lambda(A)p(B)\lambda(C)$$

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for all A, C in $\boldsymbol{B}(\boldsymbol{H})$ and B in $\boldsymbol{F}(\boldsymbol{H})$, and that p is *admissible* if it majorises λ and is majorised by τ. Thus α above is modular and admissible (the proof that τ majorises α is easy—see for example Lemma 3.8 in [**1**]).

We now give an elementary lemma which will be used frequently. Several versions of it are well known.

LEMMA 0. *Suppose A_1, A_2 are subalgebras of a Banach algebra $(\boldsymbol{B}, |\ |)$, and that A_i is a Banach algebra with respect to a norm $\|\ \|_i$ that majorises the norm $|\ |$. Then if $A_1A_2 \subseteq A_2$ there is a constant k such that*

$$\|ab\|_2 \leqq k\,\|a\|_1\,\|b\|_2, \qquad \forall\, a \in A_1,\, b \in A_2.$$

The proof is a simple application of the closed graph and uniform boundedness theorems and is omitted.

2. **The dual of A.** Schatten [**6**, Theorem 15] has characterised A'; his result is stated and proved for the case $k(A)=1$ and $\lambda \leqq \alpha \leqq \tau$ but only minor modifications are required for the general case.

Let $\boldsymbol{I}_\alpha = \{B \in \boldsymbol{B}(\boldsymbol{H}) : \alpha'(B) < \infty\}$ and let $\alpha'(B) = \sup\{|t(BF)|/\alpha(F) : 0 \neq F \in \boldsymbol{F}(\boldsymbol{H})\}$.

THEOREM 1 (SCHATTEN). *A' is linearly isometric to $(\boldsymbol{I}_\alpha, \alpha')$; $B \in \boldsymbol{I}_\alpha$ corresponds to $f_B \in A'$ where $f_B(F) = t(BF)$ for $F \in \boldsymbol{F}(\boldsymbol{H})$.*

Note that α' is an admissible norm on $\boldsymbol{F}(\boldsymbol{H})$. We recall two special cases of Theorem 1; if $\alpha=\lambda$ then $\boldsymbol{I}_\alpha = \tau c(\boldsymbol{H})$, $\alpha'=\tau$, and if $\alpha=\tau$ then $\boldsymbol{I}_\alpha = \boldsymbol{B}(\boldsymbol{H})$, $\alpha'=\lambda$.

Let $\boldsymbol{R}_\alpha = \{B \in \boldsymbol{B}(\boldsymbol{H}) : BA \in \tau c(\boldsymbol{H})\ \forall\ A \in \boldsymbol{A}\}$ and $\boldsymbol{L}_\alpha = \{B \in \boldsymbol{B}(\boldsymbol{H}) : AB \in \tau c(\boldsymbol{H})\ \forall\ A \in \boldsymbol{A}\}$. Since α and τ majorise λ it follows from the closed graph theorem that left multiplication by elements of $\boldsymbol{R}_\alpha$ and right multiplication by elements of L_α are continuous linear mappings of $\boldsymbol{A}$ into $\tau c(\boldsymbol{H})$; we denote the operator norms by $\|\ \|_\alpha$, ${}_\alpha\|\ \|$ respectively.

LEMMA 2. *$\boldsymbol{R}_\alpha = \boldsymbol{L}_\alpha$ and both are selfadjoint.*

PROOF. Let $B \in \boldsymbol{R}_\alpha$. Then from the polar representation of B we have $B^* = W^*BW^*$ for a suitable partial isometry W. So, if $F \in \boldsymbol{F}(\boldsymbol{H})$,

$$\begin{aligned}\tau(B^*F) &= \tau(W^*BW^*F)\\ &\leqq \tau(BW^*F) \leqq \|B\|_\alpha \alpha(W^*F) \leqq k(A)\,\|B\|_\alpha \alpha(F).\end{aligned}$$

Thus $B^* \in R_\alpha$, $\|B^*\|_\alpha \leqq k(A)\|B\|_\alpha$. It follows that for $F \in \boldsymbol{F}(\boldsymbol{H})$

$$\tau(FB) = \tau(B^*F^*) \leqq \|B^*\|_\alpha \alpha(F^*) \leqq k(A)\,\|B\|_\alpha \alpha(F^*)$$

which, by the α-continuity of the involution, shows that $B \in \boldsymbol{L}_\alpha$. Thus $\boldsymbol{R}_\alpha \subseteq \boldsymbol{L}_\alpha$.

Similar arguments applied to $\boldsymbol{L}_\alpha$ complete the proof.

The following characterisation of $\boldsymbol{A}'$ is essentially contained as a special case in Theorem 3.9 of [**1**] and Theorem 6.1 of [**3**].

THEOREM 3. *$\boldsymbol{A}'$ is linearly and topologically isomorphic to $(\boldsymbol{R}_\alpha, \|\ \|_\alpha)$ and to $(\boldsymbol{L}_\alpha, {}_\alpha\|\ \|)$; $B \in \boldsymbol{R}_\alpha$ corresponds to f_B where $f_B(A)=t(BA)$ for all A in $\boldsymbol{A}$.*

PROOF. Suppose $B \in \boldsymbol{R}_\alpha$; then if $F \in \boldsymbol{F}(\boldsymbol{H})$,

$$|t(BF)| \leqq \tau(BF) \leqq \|B\|_\alpha \alpha(F)$$

and hence $\alpha'(B) \leqq \|B\|_\alpha$. Thus B represents an element f_B of $\boldsymbol{A}'$ by Theorem 1 and

$$f_B(F) = t(BF), \qquad \forall\ F \in \boldsymbol{F}(\boldsymbol{H}).$$

It is easy to see that

$$f_B(A) = t(BA), \qquad \forall\ A \in \boldsymbol{A}.$$

Conversely, if $\alpha'(B) < \infty$, then for $F \in \boldsymbol{F}(\boldsymbol{H})$,

$$\begin{aligned}\tau(BF) &= \sup\{|t(BFC)|;\ C \in \boldsymbol{B}(\boldsymbol{H}),\ \lambda(C) \leqq 1\} \\ &\leqq \alpha'(B)\alpha(FC) \leqq k(A)\alpha'(B)\alpha(F).\end{aligned}$$

Thus $B \in \boldsymbol{R}_\alpha$, $\|B\|_\alpha \leqq k(A)\alpha'(B)$.

By symmetry we have the corresponding result for $(L_\alpha, {}_\alpha\|\ \|)$.

Notation. Let $\boldsymbol{R}_\alpha^0$ be the α'-completion of $\boldsymbol{F}(\boldsymbol{H})$.

LEMMA 4. *α' is modular and admissible. $\boldsymbol{R}_\alpha$ and $\boldsymbol{R}_\alpha^0$ are ideals of $\boldsymbol{B}(\boldsymbol{H})$ so that in particular they are A^*-algebras of the first kind. If $A \in \boldsymbol{A}$, $B \in \boldsymbol{B}(\boldsymbol{H})$, $C \in \boldsymbol{R}_\alpha$, then $f_{BC}(A)=f_C(AB)$, $f_{CB}(A)=f_C(BA)$.*

PROOF. Let $A \in \boldsymbol{A}$, $B \in \boldsymbol{B}(\boldsymbol{H})$, $C \in \boldsymbol{R}_\alpha$; then since $BA \in \boldsymbol{A}$, $CBA \in \tau c(\boldsymbol{H})$ and so $CB \in \boldsymbol{R}_\alpha$. Similarly $BC \in \boldsymbol{L}_\alpha$ and hence $BC \in \boldsymbol{R}_\alpha$. The equalities

$$f_{BC}(A) = f_C(AB), \qquad f_{CB}(A) = f_C(BA)$$

are immediate.

Since $\boldsymbol{R}_\alpha$ is an ideal of $\boldsymbol{B}(\boldsymbol{H})$, Lemma 0 shows that α is modular and hence $\boldsymbol{R}_\alpha^0$ is also an ideal of $\boldsymbol{B}(\boldsymbol{H})$. Finally we observe that α' is majorised by τ on $\boldsymbol{F}(\boldsymbol{H})$.

Note that $(\boldsymbol{R}_\alpha^0, \alpha')$ may now replace (A, α) in all the preceding theory. We can thus characterise $(\boldsymbol{R}_\alpha^0)'$ as $R_{\alpha'}$. We shall denote by g_T the element of $(R_\alpha^0)'$ corresponding to $T \in \boldsymbol{R}_{\alpha'}$.

3. The bidual of A.

The aim of this section will be to prove the following theorem.

THEOREM 5. *Let A'' be the bidual of $\boldsymbol{A}$ endowed with the Arens' multiplication. Then $A''=\boldsymbol{\Phi}\oplus\mathscr{R}$ where $\boldsymbol{\Phi}$ is topologically and algebraically isomorphic to $\boldsymbol{R}_{\alpha'}$, the latter having operator multiplication, and $\mathscr{R}$ is the* *radical of A'' and is linearly and topologically isomorphic to the annihilator $(R_\alpha^0)^\perp$ of R_α^0 in $(R_\alpha)'$. Further, left and right multiplication by elements of $\mathcal{R}$ annihilate A''.*

The multiplication in A'' will be one, in fact either, Arens' product. We recall the definition in stages:

$$(f * A)(B) = f(AB), \qquad A, B \in A,$$
$$(\phi * f)(A) = \phi(f * A), \qquad f \in A',$$
$$(\psi * \phi)(f) = \psi(\phi * f), \qquad \psi, \phi \in A''.$$

We shall assume subsequently that A'' has this product. A second product $\circ$ is defined in similar stages starting with the adjoint of right multiplication in A.

LEMMA 6. *If $A \neq K(H)$, then $R_{\alpha'} \subseteq K(H)$.*

PROOF If $R_{\alpha'} \nsubseteq K(H)$, then by [**2**, Proposition 10], the unit ball of R_α^0 is a bounded subset of $\tau c(H)$. Thus α' and τ are equivalent on $F(H)$, say $\alpha'(F) \geqq r\tau(F)$. Now, if $F \in F(H)$,

$$\alpha(F) = \sup\{|t(FB)| : \alpha'(B) \leqq 1, B \in R_\alpha\} = \sup\{|t(FPB)| : \alpha'(B) \leqq 1, B \in R_\alpha\}$$

where P is the orthogonal projection onto the range of F^* and so has finite rank. Thus

$$\begin{aligned}\alpha(F) &\leqq \sup\{|t(FB)| : \alpha'(B) \leqq k(R_\alpha^0), B \in F(H)\}\\ &\leqq k(R_\alpha^0)\sup\{|t(FB)| : \alpha'(B) \leqq 1, B \in F(H)\}\\ &\leqq k(R_\alpha^0)/r \cdot \sup\{|t(FB)| : \tau(B) \leqq 1, B \in F(H)\}\\ &= k(R_\alpha^0)/r \cdot \lambda(F).\end{aligned}$$

So α is equivalent to λ on $F(H)$ and hence $A = K(H)$.

LEMMA 7. *$R_{\alpha'}R_\alpha \subseteq \tau c(H)$, and for $T \in R_{\alpha'}$ the map $\tilde{g}_T$ defined by $\tilde{g}_T(B) = t(TB)$ $(B \in R_\alpha)$ is a continuous linear functional on R_α which extends g_T. The map $T \to \tilde{g}_T$ is linear and bicontinuous.*

PROOF. Let $T \in R_{\alpha'}$, $\alpha''(g_T) \leqq 1$; then $g_T \in (R_\alpha^0, \alpha')'$ and has a norm preserving extension to an element of $(R_\alpha, \alpha')'$. Since the latter is isometrically isomorphic to A'' there is, by Goldstine's theorem a net $\{A_\gamma\}_{\gamma \in \Gamma}$ in A with $\alpha(A_\gamma) \leqq 1$, $t(A_\gamma S) \to g_T(S) = t(TS)$ for all S in R_α^0.

Now for any $B \in R_\alpha$, $\{A_\gamma B\}_{\gamma \in \Gamma}$ is a bounded net in $\tau c(H)$ and so has a subnet $\{A_\gamma B\}_{\gamma \in \Gamma_0}$ which converges in the topology $\sigma(\tau c(H), K(H))$ to an element C of $\tau c(H)$. Thus for γ in Γ_0

$$t(A_\gamma BK) \to t(CK), \qquad \forall\, K \in K(H),$$

and in particular

$$(A_\gamma Bx, y) \to (Cx, y), \qquad \forall\, x, y \text{ in } \boldsymbol{H}.$$

But, for any x and y in H, $Bx \otimes y \in \boldsymbol{R}_\alpha^0$ and so, for γ in Γ,

$$t(A_\gamma Bx \otimes y) \to t(TBx \otimes y) \quad \text{i.e.} \quad (A_\gamma Bx, y) \to (TBx, y).$$

Therefore $TB = C \in \tau c(\boldsymbol{H})$.

Thus $\boldsymbol{R}_{\alpha'}\boldsymbol{R}_\alpha \subseteq \tau c(\boldsymbol{H})$, and then by Lemma 0 there is a constant s such that

$$\tau(TB) \leqq s\alpha''(T)\alpha'(B), \qquad \forall\, T \in \boldsymbol{R}_{\alpha'}, \quad B \in \boldsymbol{R}_\alpha.$$

Now for $T \in \boldsymbol{R}_{\alpha'}$, define $\tilde{g}_T$ by $\tilde{g}_T(B) = t(TB)$ ($B \in \boldsymbol{R}_\alpha$). Then $\tilde{g}_T$ is clearly an element of $(\boldsymbol{R}_\alpha)'$ with norm $\leqq s\alpha''(T)$, and $\tilde{g}_T$ extends g_T. Finally, it is easy to see that the map $T \to \tilde{g}_T$ is linear.

Notation. For $T \in \boldsymbol{R}_{\alpha'}$, let $\phi_T \in A''$ be defined by

$$\phi_T(f_S) = \tilde{g}_T(S) \qquad (S \in \boldsymbol{R}_\alpha).$$

PROOF OF THEOREM 5. If $A = \boldsymbol{K}(\boldsymbol{H})$ then $\boldsymbol{R}_\alpha = \boldsymbol{R}_\alpha^0 = \tau c(\boldsymbol{H})$ and $\boldsymbol{R}_{\alpha'} = \boldsymbol{B}(\boldsymbol{H})$ (see note following Theorem 1). It is easily verified that the Arens' product in A'' coincides with operator multiplication in $\boldsymbol{B}(\boldsymbol{H})$.

We now suppose that $A \neq \boldsymbol{K}(\boldsymbol{H})$. Let $\mathscr{R} = \{\psi \in A'' : \psi(f_S) = 0\ \forall\ S \in \boldsymbol{R}_\alpha^0\}$. Then $\mathscr{R}$ is a closed linear subspace of A''. Let $\theta \in A''$, and let $g_{T_\theta} \in (\boldsymbol{R}_\alpha^0)'$ be defined by

$$g_{T_\theta}(S) = \theta(f_S) \qquad (S \in \boldsymbol{R}_\alpha^0).$$

Now we may write $\theta = \phi_{T_\theta} + (\theta - \phi_{T_\theta})$. Let $\boldsymbol{\Phi} = \{\phi_T : T \in \boldsymbol{R}_{\alpha'}\}$; then, if $P\theta = \phi_{T_\theta}$, we have $\theta = P\theta + (\theta - P\theta)$ and thus

$$A'' = \boldsymbol{\Phi} \oplus \mathscr{R} \tag{1}$$

where the direct sum in (1) is a linear space direct sum, $\boldsymbol{\Phi}$ is linearly and topologically isomorphic to $\boldsymbol{R}_{\alpha'}$ and $\mathscr{R}$ to $(\boldsymbol{R}_\alpha^0)^\perp$.

We now examine the multiplication in A''. Let S, T denote elements of $\boldsymbol{R}_{\alpha'}$, B denote an element of $\boldsymbol{R}_\alpha$, A, C denote elements of A and ψ, ψ' denote elements of $\mathscr{R}$. Then $(f_B * A)(C) = f_B(AC) = f_{BA}(C)$, by Lemma 4. Thus

$$f_B * A = f_{BA} \tag{2}$$

and, also, $(\phi_S * f_B)(A) = \phi_S(f_B * A) = \phi_S(f_{BA}) = t(SBA) = f_{SB}(A)$, again by Lemma 4. Thus

$$\phi_S * f_B = f_{SB}. \tag{3}$$

Now $A \subseteq \boldsymbol{K}(\boldsymbol{H})$ and, by Lemma 6, $\boldsymbol{R}_{\alpha'} \subseteq \boldsymbol{K}(\boldsymbol{H})$; therefore, since α' is modular, SB and BA are in $\boldsymbol{R}_\alpha^0$. Thus $(\psi * \phi_S)(f_B) = \psi(f_{SB}) = 0$ and therefore

$$\psi * \phi_S = 0, \qquad \mathscr{R} * \boldsymbol{\Phi} = \{0\}. \tag{4}$$

Also $(\psi * f_B)(A) = \psi(f_{BA}) = 0$ and so

$$\psi * f_B = 0. \tag{5}$$

From (5) it is clear that

$$A'' * \mathscr{R} = \{0\}. \tag{6}$$

Finally $(\phi_T * \phi_S)(f_B) = \phi_T(\phi_S * f_B) = \phi_T(f_{SB}) = t(TSB) = \phi_{TS}(f_B)$. Therefore

$$\phi_T * \phi_S = \phi_{TS}. \tag{7}$$

From (7) we see that the linear isomorphism between $\boldsymbol{R}_{\alpha'}$ and $\boldsymbol{\Phi}$ is an algebra isomorphism when $\boldsymbol{R}_{\alpha'}$ has operator multiplication. Hence also $\boldsymbol{\Phi}$ is disjoint from the radical of A''. It follows from (4), (6) that $\mathscr{R}$ is the radical of A''.

Now (4), (6), (7) show that multiplication in A'' is of the required form, and that the sum (1) is an algebra direct sum. This completes the proof.

COROLLARY 8. *Modulo its radical, A'' is an A^*-algebra of the first kind. A'' is semisimple if and only if $\boldsymbol{F}(\boldsymbol{H})$ is dense in A'.*

COROLLARY 9. *A is Arens' regular.*

PROOF. All the working in §3 has been symmetrical and could equally well have been carried through for the second Arens' product. It follows that the two are equal.

Note that Garling [**2**, Theorem 8] gives an example of A for which $\boldsymbol{R}_\alpha \neq \boldsymbol{R}_\alpha^0$ and hence A'' is not semisimple.

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