ON A NONLINEAR STOCHASTIC INTEGRAL EQUATION OF THE HAMMERSTEIN TYPE

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Abstract. A nonlinear stochastic integral equation of the Hammerstein type in the form

\[ x(t; \omega) = h(t; \omega) + \int_S k(t, s; \omega)f(s, x(s; \omega)) \, d\mu(s) \]

is studied where \( t \in S \), a \( \sigma \)-finite measure space with certain properties, \( \omega \in \Omega \), the supporting set of a probability measure space \((\Omega, \mathcal{A}, P)\), and the integral is a Bochner integral. A random solution of the equation is defined to be a second order vector-valued stochastic process \( x(t; \omega) \) on \( S \) which satisfies the equation almost certainly. Using certain spaces of functions, which are spaces of second order vector-valued stochastic processes on \( S \), and fixed point theory, several theorems are proved which give conditions such that a unique random solution exists.

1. Introduction. The purpose of this note is to study the existence and uniqueness of a random solution of a nonlinear stochastic integral equation of the Hammerstein type of the form

\[ x(t; \omega) = h(t; \omega) + \int_S k(t, s; \omega)f(s, x(s; \omega)) \, d\mu(s), \]

where

(i) \( S \) is a locally compact metric space with metric \( d \) defined on \( S \times S \) and \( \mu \) is a complete \( \sigma \)-finite measure defined on the collection of Borel subsets of \( S \);

(ii) \( \omega \in \Omega \), where \( \Omega \) is the supporting set of the probability measure space \((\Omega, \mathcal{A}, P)\);

(iii) \( x(t; \omega) \) is the unknown vector-valued random variable for each \( t \in S \);

(iv) \( h(t; \omega) \) is the stochastic free term defined for \( t \in S \);

(v) \( k(t, s; \omega) \) is the stochastic kernel defined for \( t \) and \( s \) in \( S \); and

(vi) \( f(t, x) \) is a vector-valued function of \( t \in S \) and \( x \).

The integral in equation (1.1) is interpreted as a Bochner integral [12].
Further assumptions concerning the functions in (1.1) will be stated in §2.

The equation (1.1) is a generalization of stochastic integral equations studied by Padgett and Tsokos [9], Tsokos [11], and Anderson [1]. Also, equation (1.1) is a stochastic version of the deterministic integral equations which were investigated by Petryshyn and Fitzpatrick [10], Browder and Gupta [5], Browder, de Figueiredo, and Gupta [6], among others.

In order to investigate the stochastic integral equation (1.1), we will define several spaces of functions which are spaces of second order vector-valued stochastic processes on $S$ and will use certain aspects of the “theory of admissibility” of Banach spaces as introduced into the study of integral equations by Corduneanu [7] and the methods of “probabilistic functional analysis” [3].

2. Preliminaries. We will further assume that $S$ is the union of a countable family of compact subsets $\{C_n\}$ having the properties that $C_1 \subseteq C_2 \subseteq C_3 \subseteq \cdots$ and that for any other compact set in $S$ there is a $C_i$ which contains it [2].

We define $C = C(S, L_2(\Omega, A, P))$ to be the space of all continuous functions from $S$ into the space $L_2(\Omega, A, P)$ with the topology of uniform convergence on compacta. That is, for each fixed $t \in S$, $x(t; \omega)$ is a vector-valued random variable such that

$$\|x(t; \omega)\|_{L_2(\Omega, A, P)}^2 = \int_\Omega |x(t; \omega)|^2 dP(\omega) < \infty.$$  

It may be noted that $C(S, L_2(\Omega, A, P))$ is a locally convex space [12, pp. 24–26] whose topology is defined by the countable family of seminorms given by

$$\|x(t; \omega)\|_n = \sup_{t \in C_n} \|x(t; \omega)\|_{L_2(\Omega, A, P)}, \quad n = 1, 2, \ldots.$$  

Moreover, $C(S, L_2(\Omega, A, P))$ is complete relative to this topology since $L_2(\Omega, A, P)$ is complete.

We further define $BC = BC(S, L_2(\Omega, A, P))$ to be the Banach space of all bounded continuous functions from $S$ into $L_2(\Omega, A, P)$ with norm

$$\|x(t; \omega)\|_{BC} = \sup_{t \in S} \|x(t; \omega)\|_{L_2(\Omega, A, P)}.$$  

The space $BC \subseteq C$ is the space of all second order vector-valued stochastic processes defined on $S$ which are bounded and continuous in mean-square.

We will consider the functions $h(t; \omega)$ and $f(t, x(t; \omega))$ to be in the space $C(S, L_2(\Omega, A, P))$. With respect to the stochastic kernel we assume that for each pair $(t, s), k(t, s; \omega) \in L_\infty(\Omega, A, P)$ and denote the norm by

$$\|k(t, s; \omega)\|_{L_\infty(\Omega, A, P)} = P\text{-ess sup}_{\omega \in \Omega} |k(t, s; \omega)|.$$  

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Also, we will suppose that $k(t, s; \omega)$ is such that
\[
\|k(t, s; \omega)\| \cdot \|x(s; \omega)\|_{L^2(\Omega, A, P)}
\]
is $\mu$-integrable with respect to $s$ for each $t \in S$ and $x(s; \omega)$ in $C(S, L^2(\Omega, A, P))$, and that there exists a real-valued function $G$ defined $\mu$-a.e. on $S$ so that $G(s) \|x(s; \omega)\|_{L^2(\Omega, A, P)}$ is $\mu$-integrable and, for each pair $(t, s) \in S \times S$,
\[
\|k(t, u; \omega) - k(s, u; \omega)\| \cdot \|x(u; \omega)\|_{L^2(\Omega, A, P)} \leq G(u) \|x(u; \omega)\|_{L^2(\Omega, A, P)}
\]
$\mu$-a.e. Further, for almost all $s \in S$, $k(t, s; \omega)$ will be continuous in $t$ from $S$ into $L^\infty(\Omega, A, P)$.

We now define the integral operator $T$ on $C(S, L^2(\Omega, A, P))$ by
\begin{align*}
(Tx)(t; \omega) &= \int_S k(t, s; \omega) x(s; \omega) \, d\mu(s),
\end{align*}
where the integral is a Bochner integral. From the conditions on $k(t, s; \omega)$, we have that for each $t \in S$, $(Tx)(t; \omega) \in L^2(\Omega, A, P)$ and that $(Tx)(t; \omega)$ is continuous in mean square by Lebesgue's dominated convergence theorem. That is, $(Tx)(t; \omega) \in C(S, L^2(\Omega, A, P))$.

**Lemma 2.1.** The linear operator $T$ defined by equation (2.1) is continuous from $C(S, L^2(\Omega, A, P))$ into itself.

**Proof.** Note that $C(S, L^2(\Omega, A, P))$ is a Fréchet space with metric $d^*$ defined by the Fréchet combination of the sequence of seminorms $\|\cdot\|_n$, $n=1, 2, \cdots$.

Define the sequence of linear operators $\{T_M\}$, $M=1, 2, \cdots$, by
\[
(T_Mx)(t; \omega) = \int_{C_M} k(t, s; \omega) x(s; \omega) \, d\mu(s).
\]
Hence, as $M \to \infty$ we have $(T_Mx)(t; \omega) \to (Tx)(t; \omega)$.

Let $\{x_j(t; \omega)\}$ be a sequence of functions converging to $x(t; \omega)$ in $C(S, L^2(\Omega, A, P))$. Then by definition of the seminorms, for each $M$
\[
\|(T_Mx)(t; \omega) - (T_Mx_j)(t; \omega)\|_n 
\leq \sup_{t \in C_n} \int_{C_M} \|k(t, s; \omega)\| \cdot \|x(s; \omega) - x_j(s; \omega)\|_{L^2(\Omega, A, P)} \, d\mu(s).
\]
Since $\|x(s; \omega) - x_j(s; \omega)\|_{L^2(\Omega, A, P)} \to 0$ uniformly on the compact set $C_M$, for $\varepsilon > 0$ there exists a positive integer $N_M$ such that $j \geq N_M$ implies
\[
\|(T_Mx)(t; \omega) - (T_Mx_j)(t; \omega)\|_n < \varepsilon \sup_{t \in C_n} \int_{C_M} \|k(t, s; \omega)\| \, d\mu(s).
\]
Now, by the conditions on $k(t, s; \omega)$, there exists a constant $K_n$ such that $\| k(t, s; \omega) \| \leq K_n$ for all $t \in C_n$ and almost all $s$. Hence, for $j \geq N_M$

$$\| (T_M x)(t; \omega) - (T_M x_j)(t; \omega) \|_n < \varepsilon K_n \mu(C_M).$$

Since convergence in every seminorm is equivalent to convergence in the metric $d^*$, $(T_M x_j)(t; \omega)$ converges to $(T_M x)(t; \omega)$ in $C(S, L_2(\Omega, A, P))$ for each $M$. Therefore, by [8, p. 54], $T$ is continuous from $C(S, L_2(\Omega, A, P))$ into itself.

Let $B$ and $D$ be Banach spaces. The pair $(B, D)$ is said to be admissible with respect to a linear operator $T$ if $T(B) \subset D$.

**Lemma 2.2.** If $T$ is a continuous linear operator from $C(S, L_2(\Omega, A, P))$ into itself and $B, D \subset C(S, L_2(\Omega, A, P))$ are Banach spaces stronger than $C(S, L_2(\Omega, A, P))$ such that $(B, D)$ is admissible with respect to $T$, then $T$ is continuous from $B$ into $D$.

The lemma follows from the closed-graph theorem.

From Lemmas 2.1 and 2.2 it follows that $T$ defined by equation (2.1) is a bounded linear operator from $B$ into $D$.

By a random solution of the equation (1.1) we will mean a function $x(t; \omega)$ in $C(S, L_2(\Omega, A, P))$ which satisfies the equation $P$-a.e.

3. **Existence of a random solution.** We now present theorems concerning the existence and uniqueness of a random solution of the equation (1.1).

**Theorem 3.1.** We consider the stochastic integral equation (1.1) subject to the following conditions:

(i) $B$ and $D$ are Banach spaces stronger than $C(S, L_2(\Omega, A, P))$ such that $(B, D)$ is admissible with respect to the integral operator defined by equation (2.1);

(ii) $x(t; \omega) \rightarrow f(t, x(t; \omega))$ is an operator from the set

$$Q(\rho) = \{ x(t; \omega) : x(t; \omega) \in D, \| x(t; \omega) \|_D \leq \rho \}$$

into the space $B$ satisfying the Lipschitz condition

$$\| f(t, x(t; \omega)) - f(t, y(t; \omega)) \|_B \leq \lambda \| x(t; \omega) - y(t; \omega) \|_D$$

for $x(t; \omega), y(t; \omega) \in Q(\rho)$, where $\rho$ and $\lambda$ are constants;

(iii) $h(t; \omega) \in D$.

Then there exists a unique random solution of (1.1) in $Q(\rho)$, provided $\lambda K < 1$ and

$$\| h(t; \omega) \|_D + K \| f(t, 0) \|_B \leq \rho(1 - \lambda K),$$

where $K$ is the norm of $T$. 

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Proof. Define the operator $U$ from $Q(p)$ into $D$ by

$$(Ux)(t; \omega) = h(t; \omega) + \int_S k(t, s; \omega)f(s, x(s; \omega)) \, d\mu(s).$$

Then from the conditions of the theorem

$$\|(Ux)(t; \omega)\|_D \leq \|h(t; \omega)\|_D + K \|f(t, x(t; \omega))\|_B$$

$$\leq \|h(t; \omega)\|_D + K \|f(t, 0)\|_B + K\lambda \|x(t; \omega)\|_D \leq \rho.$$

Hence, $(Ux)(t; \omega) \in Q(p)$.

Now, for $x(t; \omega), y(t; \omega) \in Q(p)$ we have by condition (ii) that

$$\|(Ux)(t; \omega) - (Uy)(t; \omega)\|_D$$

$$= \left\| \int_S k(t, s; \omega)[f(s, x(s; \omega)) - f(s, y(s; \omega))] \, d\mu(s) \right\|_D$$

$$\leq K \|f(t, x(t; \omega)) - f(t, y(t; \omega))\|_B$$

$$\leq \lambda K \|x(t; \omega) - y(t; \omega)\|_D.$$

Since $\lambda K < 1$, $U$ is a contraction on $Q(p)$.

Therefore, by Banach's fixed point theorem there exists a unique $x^*(t; \omega) \in Q(p)$ which is a fixed point of $U$, that is, $x^*(t; \omega)$ is the unique random solution of equation (1.1).

A similar theorem may be obtained when $f$ is a nonlinear contraction on $Q(p)$ [4].

Theorem 3.2. Assume that the stochastic integral equation (1.1) satisfies the following conditions:

(i) same as Theorem 3.1(i);

(ii) $x(t; \omega) \rightarrow f(t, x(t; \omega))$ is an operator from the set $Q(p)$ into the space $B$ satisfying

$$\|f(t, x(t; \omega)) - f(t, y(t; \omega))\|_B \leq \phi(\|x(t; \omega) - y(t; \omega)\|_D)$$

for $x(t; \omega), y(t; \omega) \in Q(p)$, where $\phi$ is a real-valued continuous function such that $\phi(s) < s$ for $s > 0$;

(iii) $h(t; \omega) \in D$.

Then there exists a unique random solution of (1.1) in $Q(p)$, provided $K \leq 1$ and $\|h(t; \omega)\|_D + K \|f(t, 0)\|_B \leq \rho(1 - K)$, where $K$ is the norm of $T$.

The proof of Theorem 3.2 is similar to that of Theorem 3.1 except that the fixed point theorem of Boyd and Wong [4] is used.

The following is a useful application of Theorem 3.1.
Corollary 3.1. Suppose the stochastic integral equation (1.1) satisfies the following conditions:

(i) $\sup_{t \in S} \int_S \|k(t, s; \omega)\| \, d\mu(s) < \infty$;

(ii) $f(t, x)$ is a continuous function of $t \in S$ uniformly in $x$ such that for $\|x(t; \omega)\|_{BC}$, $\|y(t; \omega)\|_{BC} \leq \rho$

$$\|f(t, x(t; \omega)) - f(t, y(t; \omega))\|_{L^2(\Omega, A, P)} \leq \lambda \|x(t; \omega) - y(t; \omega)\|_{L^2(\Omega, A, P)}$$

for each $t \in S$, where $\lambda$ and $\rho$ are constants;

(iii) $h(t; \omega)$ is a bounded continuous function from $S$ into $L^2(\Omega, A, P)$.

Then there exists a unique random solution of equation (1.1), provided $\sup_{t \in S} \int_S \|k(t, s; \omega)\| \, d\mu(s)$, $\lambda$, and $\|f(t, 0)\|_{BC}$ are sufficiently small.

Proof. We must show that condition (i) implies that the pair $(BC, BC)$ is admissible with respect to the integral operator $T$ defined by equation (2.1). Let $x(t; \omega) \in BC(S, L^2(\Omega, A, P))$. Then by the properties of the Bochner integral

$$\|(Tx)(t; \omega)\|_{L^2(\Omega, A, P)} \leq \int_S \|k(t, s; \omega)x(s; \omega)\|_{L^2(\Omega, A, P)} \, d\mu(s)$$

$$\leq \sup_{t \in S} \|x(t; \omega)\|_{L^2(\Omega, A, P)} \int_S \|k(t, s; \omega)\| \, d\mu(s)$$

$$\leq \|x(t; \omega)\|_{BC} \sup_{t \in S} \int_S \|k(t, s; \omega)\| \, d\mu(s).$$

Hence, $(Tx)(t; \omega) \in BC(S, L^2(\Omega, A, P))$, that is, $(BC, BC)$ is admissible with respect to $T$.

Conditions (ii)–(iii) clearly imply that conditions (ii)–(iii) of Theorem 3.1 hold. Thus, by Theorem 3.1, there exists a unique random solution of equation (1.1).

Other corollaries of Theorems 3.1 and 3.2 may be obtained by choosing different Banach spaces contained in the space $C(S, L^2(\Omega, A, P))$ and different conditions on $f$ and $k$.

References


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