A CHARACTERIZATION OF 2-DIMENSIONAL SPHERICAL SPACE

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Abstract. The midset of two distinct points \( a \) and \( b \) of a metric space is defined as the set of all points \( x \) in the space for which the distances \( ax \) and \( bx \) are equal. A metric space is said to have the 1-WLMP if the midset of each two distinct points is a convex 1-sphere having the property that each nonmaximal (with respect to inclusion) segment intersecting it twice lies in it. We show that a nondegenerate compact space \( X \) is isometric to a 2-dimensional spherical space \( S_2(\alpha) \) (a 2-dimensional sphere of radius \( \alpha \) in euclidean 3-space with the "shorter arc" metric) if and only if \( X \) has a metric with the 1-WLMP.

Berard ([1], [2]) has given characterizations of both the 1-sphere and the 1-cell using conditions on the midsets of points in a metric space, and Buseman [4] characterized euclidean, hyperbolic, and spherical spaces among his \( G \)-spaces using convex midset properties. We characterize 2-dimensional spherical space among nontrivial compact metric spaces using a certain linear midset property described below.

The midset \( M(a, b) \) of two distinct points \( a \) and \( b \) of a metric space \( X \) is the set of all points \( x \) in \( X \) for which the distances \( ax \) and \( bx \) are equal. A metric space \( X \) is said to have the weak linear midset property (WLMP) if, for each two distinct points \( a \) and \( b \) of \( X \), the midset \( M(a, b) \) contains every nonmaximal segment (with respect to inclusion) that intersects it twice. If in addition to having the WLMP each midset in \( X \) is a convex 1-sphere we say that \( X \) has the 1-WLMP.

We prove that a nontrivial compact metric space \( X \) with the 1-WLMP is isometric to a 2-dimensional spherical space. The proof of this result is delayed until after a sequence of lemmas has been given. In each of these lemmas it is to be understood that \( X \) is a nontrivial (nondegenerate) compact metric space with the 1-WLMP. The symbol \( S(a, b) \) is used to denote a metric segment with endpoints \( a \) and \( b \), and the fact that \( q \) is between the points \( p \) and \( r \) (that is, \( p \neq q \neq r \) and \( pq + qr = pr \)) will be denoted by writing \( pqr \). A segment \( S \) is said to be maximal if it is not a proper subset of another segment. In a compact convex metric space it is well known that each two distinct points are the endpoints of at least one

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598

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segment [3, p. 41], and it is easy to show that each segment lies in a maximal one.

**Lemma 1.** If a and b are distinct points of X, then there exist points p and q in M(a, b) and a segment S(a, b) in X such that S(a, b) lies in M(p, q). In particular it follows that X is convex.

**Proof.** Let f denote a foot of a on M(a, b). Since M(a, b) is a convex simple closed curve it contains two segments S(c, f) and S(d, f) whose intersection is \{f\}. We may assume that ac \leq ad, and it follows that af \leq ac \leq ad. If equality holds in one of these inequalities, then a and b lie in a midset. This midset would contain a segment S(a, b) since it is complete and convex. In the other case where af < ac < ad, the continuous function ax, with x in S(f, d), assumes the value ac at some point q in S(f, d). If we let c = p, we have ap = aq = bq = bp, and we see that a and b are in the midset of p and q. As before, M(p, q) contains a segment S(a, b).

**Lemma 2.** If a and b are distinct points of X and S(a, b) is a nonmaximal segment, then S(a, b) is the unique segment in X having endpoints a and b.

**Proof.** Let S(a, b) and S'(a, b) be two distinct segments, and suppose that S(a, b) is properly contained in a segment S. We choose points x and y in S_1(a, b) \cup S(a, b) such that the subsegments S'_1(x, y) and S'(x, y) of S_1(a, b) and S(a, b), respectively, intersect only at their endpoints. Let m_1 and m be the midpoints of S'_1(x, y) and S'(x, y), respectively. Clearly the midset M(m_1, m) contains both x and y. Therefore it follows from the WLMP that M(m_1, m) contains S'(x, y), which contradicts the fact that m_1 \neq m_1 m.

A point q is called a ramification point of a metric space X if there exist pairwise distinct points p, r, and r' of X such that q is a midpoint of p and r and q is a midpoint of p and r'. If X is compact and convex and if X has a ramification point q, it follows that there exist two segments S(p, r) and S(p, r') both having q as a midpoint [3, p. 44].

**Lemma 3.** The space X has no ramification points.

**Proof.** Suppose that X has a ramification point q, and let S(p, r) and S(p, r') be two distinct segments with q as their midpoint. Select points x and y in the interiors of S(p, r) and S(p, r'), respectively, such that qx, qy, and qx = qy. Now both p and q lie in M(x, y); hence it follows from the WLMP that x and y belong to M(x, y). This contradiction establishes the lemma.

**Lemma 4.** If a and b are distinct points of X, then M(a, b) separates a from b in X. In fact, X - M(a, b) = A \cup B, where A = \{y \in X | ay < by\} and B = \{y \in X | ay > by\}, is the desired separation.
Lemma 5. If $a$ and $b$ are distinct points of $X$ and $u$ is a point in $M(a, b)$, then $M(a, b)$ contains a point $v$ and two maximal segments whose union is $M(a, b)$ and whose intersection is $\{u, v\}$.

Proof. Since $M(a, b)$ is a simple closed curve it contains distinct points $u$ and $t$. Since $M(a, b)$ is compact and convex, it contains a segment $S(u, t)$. Thus the partially ordered collection of all segments $S(u, x)$ (ordered by inclusion) with one endpoint $u$, containing $S(u, t)$, and lying in $M(a, b)$ has a maximal element which we call $S_1(u, v)$.

Letting $\{x_n\}$ be a sequence of points in $M(a, b) - S_1(u, v)$ converging to $v$, we see that $M(a, b)$ contains a segment $S(u, x_n)$ such that $S(u, x_n)$ and $S_1(u, v)$ have only the point $u$ in common. A positive integer $N$ exists such that $ux_n \in x_n$ holds for $n > N$; thus $ux_n + x_n = ux_n$. By the continuity of the metric it follows that $ux_n v$ holds, and since $M(a, b)$ is compact and convex, it contains two segments $S(u, x_1)$ and $S(x_1, v)$ whose union is a segment $S_2(u, v)$ [3, p. 44]. It is clear that $S_1(u, v) \cup S_2(u, v) = M(a, b)$ since $M(a, b)$ is a simple closed curve, and from the construction of $S_2(u, v)$ we see that $S_1(u, v) \cap S_2(u, v) = \{u, v\}$. If $S_2(u, v)$ were not maximal it would follow from the WLMP that $M(a, b)$ would contain a point $e$ such that either $uve$ or $vue$ holds, contrary to the fact that $uev$ holds since $e$ would lie in $S_1(u, v)$.

Definition. Let $a$ and $b$ be distinct points of $X$. The cone on $M(a, b)$ from $a$ is the union of all segments $S(a, y)$ where $y$ lies in $M(a, b)$.

Lemma 6. Let $a$ and $b$ be distinct points of $X$. If for each point $y$ in $M(a, b)$ there is a unique segment with endpoints $a$ and $y$, then the cone on $M(a, b)$ from $a$ is a 2-cell.

Proof. We first note that if $x$ and $y$ are distinct points of $M(a, b)$, then $S(a, x)$ and $S(a, y)$ have only the point $a$ in common. Otherwise $X$ would either contain a ramification point (contrary to Lemma 3) or the WLMP would imply that $a$ lies in $M(a, b)$ (contrary to $aa \neq ab$).

Let $S$ denote the circle $\{(x, y, 0)| x^2 + y^2 = 1\}$ in $E^3$, and let $f$ be a homeomorphism from $M(a, b)$ onto $S$. We denote $f(y)$ by $y'$, and we define $a' = f(a)$ to be the point $(0, 0, 1)$ in $E^3$. We extend $f$ to a homeomorphism from the cone $C$ on $M(a, b)$ from $a$ onto the cone $C'$ on $S$ from $a'$ as follows. For each $x$ in $C - M(a, b) - \{a\}$ there is a unique point $y$ such that $axy$ holds. Let $x' = f(x)$ be that point on $S(a', y')$ such that $a'x'y' = ax/ay$. From above it is clear that $f$ is a bijection. If $\{x_n\}$ is a sequence of points of $C$ converging to $x_0$, and, for each $i$, $y_i$ is the point of $M(a, b)$ such that $x_i$ belongs to $S(a, y_i)$, then it follows that $\{y_n\}$ converges to $y_0$. Thus $\{y'_n\}$ converges to $y'_0$, and then $\{x'_n\}$ must converge to $x'_0$. In the same
manner we see that $f^{-1}$ is continuous; hence $f$ is a homeomorphism. Since $C'$ is known to be a 2-cell, the lemma follows.

**Theorem 1.** A nondegenerate compact space $X$ is a 2-sphere if and only if there is a metric for $X$ under which $X$ has the WLMP.

**Proof.** If $X$ is a 2-sphere, then it is homeomorphic to

$$\{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$$

in $E^3$. The usual "shortest arc" metric on this round sphere satisfies the conditions of the theorem. For the other half of the proof we assume that $X$ is a nondegenerate compact metric space with the 1-WLMP.

Let $a'$ and $b'$ be two distinct points of $X$, and let $u$ be in $M(a', b')$. According to Lemma 5 there exists another point $v$ in $M(a', b')$ and two maximal segments whose union is $M(a', b')$ and whose intersection is $\{u, v\}$. If $m$ and $m'$ are the midpoints of these maximal segments, then $m$ and $m'$ belong to the convex compact set $M(u, v)$. Then $M(u, v)$ contains a segment $S(m, m')$, and this segment is maximal for otherwise either $u$ or $v$ would belong to $M(u, v)$ by the WLMP. From Lemma 5 there is another maximal segment in $M(u, v)$ joining $m$ to $m'$. We let $a$ and $b$ be the midpoints of these two maximal segments in $M(u, v)$. Note that $m$ and $m'$ belong to $M(a, b)$, and since $a$ and $b$ lie in $M(u, v)$ we know that $ua=av$ and $bu= bv$. Now $a$, $b$, $u$, and $v$ all lie in $M(m, m')$, and from the WLMP we see that any segment in $M(m, m')$ joining $u$ and $v$ is maximal. Since two such maximal segments exist in $M(m, m')$ we know that $uav$ and $ubv$ hold. Thus $2(au)=uv$ and $2(bu)=uv$. Thus $au=bu$, and similarly $av=bv$. This means that $u$ and $v$ belong to $M(a, b)$. Since $m$, $m'$, $u$, and $v$ all lie in $M(a, b)$, it follows from the WLMP that $M(a, b)=M(a', b')$. We have no more need for $a'$ and $b'$.

We now show that for each $y$ in $M(a, b)$ there is a unique segment with endpoints $a$ and $y$. If $y$ is either $m$ or $m'$, then a nonmaximal segment $S(a, y)$ exists in $M(a, b)$, and therefore $S(a, y)$ is unique by Lemma 2. Since $M(m, m')$ contains $a$, $b$, $u$, and $v$ it follows from the WLMP that there is a nonmaximal segment $S(a, y)$ in $M(m, m')$ if $y$ is either $u$ or $v$. Thus such a segment is also unique. We now assume that $y \notin \{m, m', u, v\}$. By relabeling points if necessary it may be assumed that $myu$ holds. From Lemma 1 we know that there exists a segment $S(a, y)$ and two distinct points $p$ and $q$ in $M(a, y)$ such that $M(p, q)$ contains $S(a, y)$. Since the segment $S(a, y)$ is known to be unique unless it is a maximal segment (by Lemma 2), we suppose that $S(a, y)$ is maximal in order to obtain a contradiction. It follows from Lemma 5 that $M(p, q)$ is the union of two maximal segments $S_1(a, y)$ and $S(a, y)$ whose intersection is $\{a, y\}$. Notice that $M(p, q) \cap M(a, b) = \{y\}$, for otherwise the 1-WLMP implies
the contradiction that $a$ belongs to $M(a, b)$. Let $r$ and $n$ be the midpoints of $S(a, y)$ and $S_1(a, y)$, respectively, and note that $M(a, y)$ contains \{r, n\}. Thus, from Lemma 5, $M(a, y)$ is the union of two segments $S_1$ and $S_2$, having endpoints $r$ and $n$, such that $S_1 \cap S_2 = \{r, n\}$. Let $A$ and $B$ be the mutually separated sets promised by Lemma 4 whose union is $X - M(a, b)$ with $a \in A$ and $b \in B$. Notice that both $r$ and $n$ belong to $A$. Suppose that there is a point $f$ of $B$ in one of the two segments $S_1$ and $S_2$. Then $M(a, b)$ would contain two interior points of $S_i$, $i = 1$ or $2$, since the endpoints of $S_i$ both lie in $A$. From WLMP it would follow that $f$ lies in $M(a, b)$, a contradiction. Thus $M(a, y) \cap B = \emptyset$. Since $M(a, y)$ separates $a$ from $y$ it must follow that $M(a, y)$ intersects each of the unique segments $S(a, u)$, $S(a, m)$, $S(a, m')$, and $S(a, v)$. Since $S(a, u) \cup S(a, v)$ is a maximal segment in $M(m, m')$ (see Lemma 5 and the WLMP), $M(a, y)$ cannot intersect its interior twice. Then $u$ and $v$ must lie in $M(a, y)$. Similar reasoning shows that both $m$ and $m'$ lie in $M(a, y)$. From the 1-WLMP it follows that $M(a, b)$ lies in $M(a, y)$, contrary to the fact that $y$ does not belong to $M(a, y)$. Therefore the segment $S(a, y)$ is unique. Similarly segments $S(b, y)$, with $y$ in $M(a, b)$, are unique.

Now we may apply Lemma 6 to obtain two 2-cells $D_1$ and $D_2$ by coning $M(a, b)$ from $a$ and $b$, respectively. We shall prove that the 2-sphere $X' = D_1 \cup D_2$ is $X$. Suppose to the contrary that there is a point $y$ in $X - X'$. We may suppose that $ay < by$ since $y \notin M(a, b)$. Let $S(a, y)$ be a segment and notice that it does not intersect $M(a, b)$. Since $X$ has no ramification points (Lemma 3) it follows that $S(a, y) \cap X' = \{a\}$. Choose a point $z$ in $M(a, b)$, and a segment $S(y, z)$. Order $S(y, z)$ from $y$ to $z$ and pick the first point $y'$ of $X' \cap S(y, z)$. Since $ay < by$ and $S(y, y') \cap M(a, b)$ contains at most the point $y'$, it follows that $y'$ lies in $D_1$. Let $p$ be the point of $M(a, b)$ such that $y' \in S(a, p)$. Now $M(a, y')$ cannot intersect the segment $S(a, p)$ at the point other than the midpoint $t$ of $S(a, y')$ by the WLMP. Thus there is a segment $S(h, k)$ in $M(a, b)$, with $p$ in its interior, such that $S(h, k) \cap M(a, y') = \emptyset$. The cone $D$ on $S(h, k)$ from $a$ is a 2-cell (see the proof of Lemma 6) in $D_1$. Each segment $S(a, x)$, with $x$ in $S(h, k)$, must intersect $M(a, y')$, for otherwise $M(a, y')$ would not separate $a$ from $y'$ in $D$. From WLMP the intersection of $S(a, x)$ with $M(a, y')$ consists of a single point $w_x$. Let $R$ be the union of all segments $S(a, w_x)$, $x$ in $S(h, k)$, and let $T$ be the union of all segments $S(w_x, x)$. Then $R \cup T = D$ and $R$ and $T$ are each closed and connected. From the unicoherence of $D$ [5, p. 374], $R \cap T$ is connected. Since $R \cap T \subset M(a, y')$, there must be an arc $G$ in $R \cap T$ with $t$ in its interior.

Let $\{x_i\}$ be a sequence of points in $S(y, y')$ converging to $y'$, and notice that there is an integer $K$ such that, for $i > K$, each segment $S(a, x_i)$ intersects $M(a, y')$ at a unique point $t_i$. This is because $M(a, y')$ separates
a from $y'$ in the simple closed curve in $S(a, x_i) \cup S(x_i, y') \cup S(a, y')$ and, for large $i$, $S(x_i, y') \cap M(a, y') = \emptyset$; the uniqueness of $t_i$ comes from the WLMP. The continuity of the metric insures that $\{t_i\}$ converges to $t$. This is a contradiction since it is impossible for the simple closed curve $M(a, y')$ to contain an arc $G$ and to contain a sequence $\{t_i\}$ of points not in $G$ but converging to the interior point $t$ of $G$.

This establishes the fact that $X$ is the 2-sphere $X'$, and completes the proof of Theorem 1.

**Theorem 2.** A nontrivial compact space $X$ is isometric to a 2-dimensional spherical space if and only if $X$ has a metric with the 1-WLMP.

**Proof.** Since 2-dimensional spherical space has a metric with the 1-WLMP we need only show the proof in the other direction. Thus we now assume that $X$ is a nontrivial compact metric space with the 1-WLMP. From Theorem 1, $X$ is homeomorphic to a 2-sphere.

Busemann [4] has shown that a 2-dimensional compact G-space with convex midsets is isometric with 2-dimensional spherical space. Since $X$ is 2-dimensional, compact, and has convex midsets, Theorem 2 will follow from [4] once we show that $X$ is a G-space. The only condition on a G-space that is not either obvious or given by previous lemmas is the locally externally convex property. We shall now show that $X$ has this property.

We assume all of the proof and notation from Theorem 1 up to the last two paragraphs, so that we know the 2-sphere $X$ is the union of the two 2-cells $D_1$ and $D_2$. These cells are the cones from $a$ and $b$, respectively, on $M(a, b)$. A point $p$ of $X$ cannot lie in all three midsets $M(a, b)$, $M(u, v)$, and $M(m, m)$, so we assume for convenience that $p$ is not in $M(a, b)$. A connected neighborhood $N$ of $p$ is chosen so that $N \cap M(a, b) = \emptyset$. Let $x$ and $y$ be two points of $N$. To show that $X$ is locally externally convex at $p$ it suffices to exhibit a point $z$ in $N$ such that $xyz$. From Lemma 1 we see that there exist two points $q$ and $r$ and a segment $S$ joining $x$ and $y$ such that $S \subseteq M(q, r)$. The selection of $z$ can be made in $M(q, r)$ if $S$ is not a maximal segment. Suppose that $S$ is a maximal segment in the simple closed curve $M(q, r)$. Since $N$ does not intersect $M(a, b)$, we may assume that $x$ and $y$ are both closer to $a$ than to $b$. From the WLMP it follows that $S$ lies in $D_1$. (Notice that if we had assumed that $p$ was not in some midset other than $M(a, b)$ at the outset, then we could still go through the proof of Theorem 1 to write $X$ as the union of two 2-cells $D'_1$ and $D'_1$, each a cone on a midset. Thus the proof would go through just the same.) Since $M(q, r)$ is the union of two maximal segments $S$ and $S'$ with endpoints $x$ and $y$ and since $S$ lies in $D_1$, it follows from the WLMP that $M(q, r)$ lies in $D_1$. Unless $M(q, r)$ separates $a$ from $M(a, b)$, some segment from $a$ to $M(a, b)$ would intersect $M(q, r)$ twice and would consequently...
lie in $M(q, r)$. This would force $N$ to contain a point of $M(a, b)$. Thus $M(q, r)$ separates $a$ from $M(a, b)$. But then a segment $S(m, m')$ in $M(u, v)$ would intersect $M(q, r)$ twice (at least once in its interior) and would lie in $M(q, r)$ by the WLMP. Again this would contradict the fact that $N$ does not intersect $M(a, b)$.

Thus $X$ is locally externally convex and the proof is complete.

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