PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 38, Number 3, May 1973

DEGREES OF NONRECURSIVE PRESENTABILITY

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ABSTRACT. We prove that every $\mathbf{0}'$-recursive binary relation of natural numbers is isomorphic to a recursive relation restricted to a co-r.e. subset of its domain. We use this result to define and discuss "degrees of nonrecursive presentability."

1. **Introduction.** We assume that the reader is familiar with the notation and terminology of [**6**]. A denumerable structure is said to be *recursive* (*X-recursive*, *r.e.*, Π_1^0, *tt-reducible to X*, *etc.*) if and only if its universe is a recursive (X-recursive, r.e., Π_1^0, *tt-reducible to* X, etc.) subset of the natural numbers, and its relations and operations[1] are recursive (X-recursive, r.e., Π_1^0, tt-reducible to X, etc.). Let $\mathscr{K}$ be a class of structures. One might wish to discuss the Turing (tt, many-one) degrees of those members of $\mathscr{K}$ which have no recursive isomorph. It turns out that such a discussion, for the cases $\mathscr{K}$=linear orderings, and $\mathscr{K}$=Boolean algebras, respectively, leads to a solution of two homogeneity problems; namely it shows that there is no jump-preserving isomorphism from {Turing degrees $\geqq \mathbf{0}^{(6)}$} to {Turing degrees}, and there is no isomorphism from the lattice of ϕ'-r.e. sets to the lattice of r.e. sets. (See [**1**] and [**2**].)

In this paper, we prove some results for the general case. Toward this end, we define the following partial orderings on 2^N. If $\mathscr{K}$ is a class of denumerable structures of a given finite similarity type, and if X, $Y \subseteq N$, then we write $X \leqq_{\approx T}(\mathscr{K})\, Y$, $(X \leqq_{\approx tt}(\mathscr{K})\, Y$, $X \leqq_{\approx m}(\mathscr{K})\, Y)$ if and only if every X-recursive (tt-reducible to X, many-one reducible to X) member of $\mathscr{K}$ has a Y-recursive (tt-reducible to Y, many-one reducible to Y) isomorph. In this paper, we show that, for any class $\mathscr{K}$, as defined above,

(i) $\leqq_{tt} \subsetneqq \leqq_{\approx tt}(\mathscr{K})$,

(ii) $\leqq_m \subsetneqq \leqq_{\approx m}(\mathscr{K})$; and, for $\mathscr{K}$=denumerable groups, denumerable fields, permutations of N, denumerable binary relations,

(iii) $\leqq_{\approx T}(\mathscr{K}) = \leqq_T$.

Received by the editors December 7, 1969 and, in revised form, November 12, 1970.

AMS (MOS) subject classifications (1970). Primary 02F30, 02F35; Secondary 02H05.

Key words and phrases. Recursive function, arithmetic hierachy, degree of unsolvability, theory, structure.

[1] If $f: N^2 \to N$ is a function, we say that f is, for example, Π_1^0 if and only if $\{\langle x, y, z\rangle | z = f(x, y)\}$ is Π_1^0.

We actually prove the results for the case $\mathscr{K}=\{$denumerable binary relations$\}$ but the proofs easily generalize.

Notation. Let $\boldsymbol{B}$ be the theory of one binary relation. Let

$$\leqq_{\approx T}(\leqq_{\approx tt}, \leqq_{\approx m}) \text{ be } \leqq_{\approx T}(\mathscr{K})(\leqq_{\approx tt}(\mathscr{K}), \leqq_{\approx m}(\mathscr{K})),$$

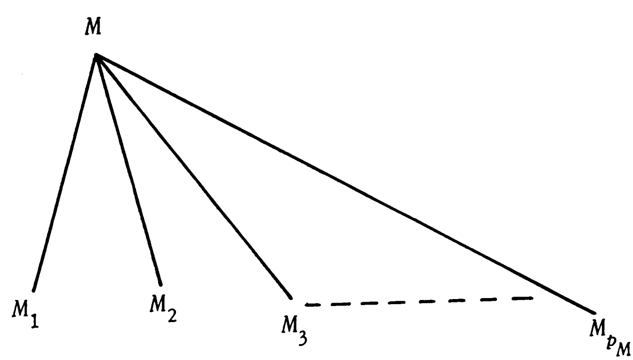
where $\mathscr{K}=\{$denumerable models of $\boldsymbol{B}\}$.

2. **A universal binary relation.** Let R be a binary relation and B be a subset of the universe of R. Then $R\restriction B$ will denote the restriction of R to B. If $M=\{A, R\}$ is a model of $\boldsymbol{B}$ and $B\subseteq A$, then $M\restriction B=\{B, R\restriction B\}$. If $\{M_\mu\}_{\mu\in I}=\{A_\mu, R_\mu\}_{\mu\in I}$ is a collection of models of $\boldsymbol{B}$, then

$$\bigcup_\mu M_\mu = \left\{\bigcup_\mu A_\mu, \bigcup_\mu R_\mu\right\}.$$

If $M_1=M_2\restriction A$, for some set A, then we write $M_1\subseteq M_2$. If $M\subseteq M_1$, and $M\subseteq M_2$, then $M_1\cong_M M_2$ means that there is an isomorphism from M_1 to M_2 which is the identity on the universe of M.

We now define a tree, $\mathscr{T}$, each of whose vertices is a finite model of $\boldsymbol{B}$ and whose branches are linearly ordered by $\subseteq$. The initial vertex of $\mathscr{T}$ is the empty model. If M is a vertex of $\mathscr{T}$ then we have the following situation:



where $M_1, \cdots, M_{p_M}$ are all the one element extensions of M. (Where two one-element extensions M' and M'' of M are identified if and only if $M'\cong_M M''$.)

Let $\mathscr{U}=\bigcup_{M\in\mathscr{T}} M$.

THEOREM 2.1. *Every denumerable model of $\boldsymbol{B}$ is embeddable in $\mathscr{U}$.*

PROOF. Let $M=\{\{a_1, a_2, a_3, \cdots\}, R\}$ be a denumerable model of $\boldsymbol{B}$. Let $M_n=M\restriction\{a_1,\cdots, a_n\}$. The sequence $\{M_1, M_2, M_3, \cdots\}$ corresponds, in an obvious fashion, to a branch, β, of $\mathscr{T}$. It is easily seen that M is isomorphic to $\bigcup_{M\in\beta} M \subseteq \mathscr{U}$. $\square$

It is clear that, despite its highly nonconstructive definition, $\mathscr{U} \cong \{N, R_{\mathscr{U}}\}$ where $R_{\mathscr{U}}$ is recursive, and, where $\{N, R_{\mathscr{U}}\}$ is "constructively universal" in the following sense.

COROLLARY 2.2. *If $X \subseteq N$, then any X-recursive model of $\boldsymbol{B}$ is X-recursively embeddable in $\{N, R_{\mathscr{U}}\}$.*

3. $\leqq_{\approx tt}$.

LEMMA 3.1. *Any Σ_2^0 subset of N can be expressed in the form*

$$\{n \mid (\exists!\, x)(\forall\, y)Q(x, y, n)\}$$

where $Q(x, y, n)$ is recursive.

PROOF. Straightforward.

THEOREM 3.2. *Any ϕ'-recursive model*[2] *of $\boldsymbol{B}$ is isomorphic to a Π_1^0 model.*[3]

PROOF. Let $M = \{A, R\}$ be a ϕ'-recursive model of $\boldsymbol{B}$.

Let $f: M \to \{N, R_{\mathscr{U}}\}$ be a ϕ'-recursive embedding. (f exists by 2.2.) The Range of f is Σ_2^0, and, by 3.1, Range $(f) = \{n \mid (\exists!\, x)(\forall y)Q(x, y, n)\}$, where $Q(x, y, n)$ is recursive. Let $\mathscr{U}' = \{N^2, R_{\mathscr{U}'}\}$ where

$$\langle\langle s, t\rangle, \langle s', t'\rangle\rangle \in R_{\mathscr{U}'} \Leftrightarrow_{\text{df.}} \langle s, s'\rangle \in R_{\mathscr{U}}.$$

Let $B \subseteq N^2$ be $\{\langle s, t\rangle \mid (\forall y)Q(t, y, s)\}$. B is Π_1^0. Finally, if $z \in A$, let $g(z)$ be the unique $\langle s, t\rangle$ such that $s = f(z)$ and $\langle s, t\rangle \in B$. $g: M \to \{B, R_{\mathscr{U}'} \restriction B\}$. Since f is 1-1, so is g. g is onto by 3.1. By the definition of $\{N^2, R_{\mathscr{U}'}\}$, g is a homomorphism. To complete the proof, we observe that

$$\{B, R_{\mathscr{U}'} \restriction B\} \text{ is a } \Pi_1^0 \text{ model of } \boldsymbol{B}.$$

COROLLARY 3.3. (i) $\leqq_{\approx tt} \neq \leqq_{tt}$, (ii) $\leqq_{\approx m} \neq \leqq_m$.

PROOF. By 3.2 and [**6**, 82] every ϕ'-recursive set is $\leqq_{\approx tt} N - \phi'$. (i) now follows from the existence of an ϕ'-recursive set which is not tt-reducible to $N - \phi'$, [**6**, p. 127, Theorem I]. (ii) is proved similarly.

4. $\leqq_{\approx T}$.

THEOREM 4.1. *If $X, Y \subseteq N$, then $X \leqq_T Y$ if and only if $X \leqq_{\approx T} Y$.*

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[2] X' is the jump of X, $\varnothing$ is the empty set.

[3] If we require that the universe of a Π_1^0 model be N, then this theorem is false [**5**]. However, it seems natural to allow the universe of a model to be a proper subset of N, so that one could, for example, say that ω_1^{Kleene} is isomorphic to a Π_1^1 linear ordering [**4**].

PROOF. $\Rightarrow$. Obvious. Suppose $X \leqq_{\approx T} Y$. Let $\boldsymbol{P}$ be the theory of one permutation. Let M_X be an X-recursive model of $\boldsymbol{P}$ such that M_X has cycle of size $2n$ iff $n \in X$, and M_X has a cycle of size $2n+1$ iff $n \notin X$. It is easily seen that X is recursive in any isomorph of M_X. Since M_X has a Y-recursive isomorph, $X \leqq_T Y$. $\square$

5. Concluding remarks.

REMARK 1. One can use the methods of 3.2 to show that any ϕ'-recursive linear ordering is isomorphic to a Π_1^0 subset of the rational numbers.

REMARK 2. It turns out that for $\mathscr{K}$=linear orderings [**1**], Boolean algebras [**2**], equivalence relations [**3**], rings, groups, fields ϕ' non-$\leqq_{\approx T}(\mathscr{K})\phi$. The only general result we have been able to obtain along these lines is the following:

HIERARCHY THEOREM. *If $\mathscr{K}$ is an arithmetically axiomitizable class of models, and if there is an infinite, arithmetic set, S, of sentences such that $Th(\mathscr{K}) \cup S'$ is consistent where S' contains ψ or $\neg\psi$ for each $\psi \in S$, then $(\forall n)(\phi^{(\omega)}$ non-$\leqq_{\approx T}(\mathscr{K})\phi^{(n)})$. (Where $\phi^{(\omega)}=\{\langle x, n\rangle | x \in \phi^{(n)}\}$.)*

PROOF (unpublished). A diagonalization argument very similar to [**2**, §3] together with a Henkin procedure. $\square$

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