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DIMENSION THEORY

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ABSTRACT. The T-equivalence and T-iterations introduced for a bifunctor T enable one to deduce a "shifting property" from which one gets the "long exact sequence of homology." Also, the appropriate lemma of Schanuel is proved, from which one can develop a T-dimension theory. These notions are useful in proving known duality homomorphisms and may serve to get some new ones. For one purpose, they unify the methods of studying the various common dimensions.

1. Let $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ be abelian categories. Let $T(A, B)$ be an additive bifunctor from $\boldsymbol{A}\times\boldsymbol{B}$ to $\boldsymbol{C}$. Suppose that T is covariant in both variables and that $T(A, -)$ and $T(-, B)$ are left exact functors respectively. Suppose that for every object A in $\boldsymbol{A}$ there exists an object A' in $\boldsymbol{A}$ such that there exists an epimorphism from A' onto A and such that $T(A', -)$ is an exact functor. A similar hypothesis is presumed on $\boldsymbol{B}$. $\mathscr{P}(\boldsymbol{A})$ ($\mathscr{P}(\boldsymbol{B})$) denotes the objects in $\boldsymbol{A}(\boldsymbol{B})$ for which $T(A, -)$ ($T(-B)$) is exact.

The definitions for $\boldsymbol{B}$ are similar to those for $\boldsymbol{A}$:

(i) A, A' in $\boldsymbol{A}$ are T-related if there exists an exact sequence in $\boldsymbol{A}$: $0\to U\to V\to W\to 0$ such that $W\in\mathscr{P}(\boldsymbol{A})$ and $\{U, V\}=\{A, A'\}$.

(ii) A, A' in $\boldsymbol{A}$ are T-equivalent if there exists a chain in $\boldsymbol{A}$, $A=A_1$, $\cdots$, $A_n=A'$ such that A_i, A_{i+1} are T-related for $i=1, \cdots, n-1$.

REMARK. $T(0, B)=0$, $T(A, 0)=0$ since T is left exact; thus $0\in\mathscr{P}(\boldsymbol{A})$ and $0\in\mathscr{P}(\boldsymbol{B})$.

The T-equivalence is an equivalence relation.

(iii) A_1 is a first T-iteration of A if there exists an exact sequence in $\boldsymbol{A}$: $0\to A_1\to W\to A\to 0$ where $W\in\mathscr{P}(\boldsymbol{A})$. A_{n+1} is a $(n+1)$st T-iteration of A if it is a first T-iteration of A_n.

THE LEMMA OF SCHANUEL. *If A is T-equivalent to A', then A_1 is T-equivalent to A_1'. In particular, any pair of first T-iterations for A are T-equivalent.*

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Consider the exact sequences:

$$0 \longrightarrow A_1 \longrightarrow W \xrightarrow{i} A \longrightarrow 0, \qquad 0 \longrightarrow A_1' \longrightarrow W' \xrightarrow{j} A' \longrightarrow 0,$$

where A and A' are T-equivalent and W, $W' \in \mathscr{P}(\boldsymbol{A})$. The proof will follow in a sequence of steps:

LEMMA 1. *In an exact sequence* $0 \to W' \to W \to W'' \to 0$ *in* $\boldsymbol{A}$, *if* $W'' \in \mathscr{P}(\boldsymbol{A})$ *then* $W \in \mathscr{P}(\boldsymbol{A})$ *iff* $W' \in \mathscr{P}(\boldsymbol{A})$.

PROOF. For any exact sequence $0 \to B' \to B \to B'' \to 0$ in $\boldsymbol{B}$ there results a commutative square:

$$\begin{array}{ccc} T(W, B) & \xrightarrow{g} & T(W, B'') \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ T(W'', B) & \xrightarrow{h} & T(W'', B'') \end{array}$$

In particular, if $B \in \mathscr{P}(\boldsymbol{B})$ then α is an epimorphism. Since $W'' \in \mathscr{P}(\boldsymbol{A})$, then h is an epimorphism. Consequently β is an epimorphism. We may thus conclude that the sequence $0 \to T(W', B'') \to T(W, B'') \to T(W'', B'') \to 0$ is exact for all $B'' \in \boldsymbol{B}$. In particular, there results the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T(W', B') & \longrightarrow & T(W', B) & \xrightarrow{f} & T(W', B'') & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T(W, B') & \longrightarrow & T(W, B) & \xrightarrow{g} & T(W, B'') & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T(W'', B') & \longrightarrow & T(W'', B) & \xrightarrow{h} & T(W'', B'') & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

from which it results (e.g. the 3×3 Lemma) that f is an epimorphism iff g is an epimorphism.

COROLLARY 2. *Let* $0 \to A \to A' \to W'' \to 0$ *be an exact sequence, with* $W'' \in \mathscr{P}(\boldsymbol{A})$, *and let* $W' \in \mathscr{P}(\boldsymbol{A})$ *be such that there exists an epimorphism from* W' *onto* A'. *Then the kernel* W *of the induced epimorphism* $W' \to W''$ *is an object*

of $\mathscr{P}(A)$ and there exists a commutative diagram with exact rows and columns:

$$\begin{array}{ccccc}
0 & & 0 & & \\
\downarrow & & \downarrow & & \\
W & \to & A & \to & 0 \\
\downarrow & & \downarrow & & \\
W' & \to & A' & \to & 0 \\
\downarrow & & \downarrow & & \\
W'' & = & W'' & & \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}$$

COROLLARY 3. *Let $0 \to A' \to A \to A'' \to 0$ be an exact sequence. Then a commutative diagram exists with exact rows and columns*:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A_1' & \to & A_1 & \to & A_1'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & W' & \to & W & \to & W'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A' & \to & A & \to & A'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with W', W, W'' in $\mathscr{P}(A)$.

PROOF. For instance, pick any suitable W', pick W'' from which there is an epimorphism onto A, and let W be $W' \oplus W''$. With the natural homomorphisms one deduces the diagram as stated.

PROOF OF THE LEMMA OF SCHANUEL. By Corollary 2 it suffices to prove the lemma for the case $A = A'$. Consider this case and let U be the graph of i and j. There results a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & A_1 \arrow[r, equal] \arrow[d] & A_1 \arrow[d] & \\
0 \arrow[r] & A_1' \arrow[r] \arrow[d, equal] & U \arrow[r] \arrow[d] & W \arrow[r] \arrow[d, "i"] & 0 \\
0 \arrow[r] & A_1' \arrow[r] & W' \arrow[r, "j"] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Then A_1, U and A_1', U are T-related, and therefore A_1, A_1' are T-equivalent.

COROLLARY 4. *The T-equivalence class of the nth iteration depends solely upon the T-equivalent class of the object whose iterations are under consideration.*

At this point one can discuss the T-dimension, as, for instance, the comparison of dimensions of A, A', A'' whenever $0\to A'\to A\to A''\to 0$ is an exact sequence.

Our next object is the satellites and their long exact sequences.

Let $W\in\mathscr{P}(\boldsymbol{A})$, $M\in\mathscr{P}(\boldsymbol{B})$ and consider the exact sequences $0\to A_1\to W\to A\to 0$, $0\to B_1\to M\to B\to 0$. There results a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T(A_1, B_1) \arrow[r] \arrow[d] & T(A_1, M) \arrow[r] \arrow[d] & T(A_1, B) \arrow[r] \arrow[d] & X \arrow[r] & 0 \\
0 \arrow[r] & T(W_1, B_1) \arrow[r] \arrow[d] & T(W_1, M) \arrow[r] \arrow[d] & T(W_1, B) \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & T(A, B_1) \arrow[r] \arrow[d] & T(A_1, M) \arrow[r] \arrow[d] & T(A_1, B) \arrow[r] \arrow[d] & Z \arrow[r] & 0 \\
 & Y \arrow[d] & 0 & V \arrow[d] & & \\
 & 0 & & 0 & &
\end{tikzcd}$$

Then $X=Y$ and $Z=V$. However, Z as defined seems to depend upon M and V depends upon W, that is, Z is a "function" of A, B and M, while V is a "function" of A, B and W. Since $Z=V$, they must be independent both of M and W. Set $Z=E_1(A, B)$. Consequently, $X=E_1(A_1, B)=E_1(A, B_1)=Y=E_2(A, B)$ is well defined. An easy matter is now to check that $E_n(A, B)$ is well defined as $E_i(A_j, B_k)$ for $i+j+k=n$, where $A_0=A$, $B_0=B$.

The elements $E_i(A, B)$ fit in a long exact sequence as expected: for $0\to A'\to A\to A''\to 0$ there results a commutative diagram with exact rows and columns for every integer n, where W_n, W_n', $W_n''\in\mathscr{P}(\boldsymbol{A})$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_{n+1}' \arrow[r] \arrow[d] & A_{n+1} \arrow[r] \arrow[d] & A_{n+1}'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W_{n+1}' \arrow[r] \arrow[d] & W_{n+1} \arrow[r] \arrow[d] & W_{n+1}'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_n' \arrow[r] \arrow[d] & A_n \arrow[r] \arrow[d] & A_n'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

from which there results the commutative diagram with exact rows and columns:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
(1) & 0 \arrow[r] & T(A'_{n+1}, B) \arrow[r] \arrow[d] & T(A_{n+1}, B) \arrow[r] \arrow[d] & T(A''_{n+1}, B) \arrow[d] & \\
(2) & 0 \arrow[r] & T(W'_{n+1}, B) \arrow[r] \arrow[d] & T(W_{n+1}, B) \arrow[r] \arrow[d] & T(W''_{n+1}, B) \arrow[r] \arrow[d] & 0 \\
(3) & 0 \arrow[r] & T(A'_n, B) \arrow[r] \arrow[d] & T(A_n, B) \arrow[r] \arrow[d] & T(A''_n, B) \arrow[r, dashed, "a"] \arrow[d] & {[E_n(A', B) \to E_n(A, B) \to \cdots]} \\
(4) & & E_{n+1}(A', B) \arrow[r] \arrow[d] & E_{n+1}(A, B) \arrow[r] \arrow[d] & E_{n+1}(A'', B) \arrow[ur, "\Delta"] \arrow[d] & \\
 & & 0 & 0 & 0 &
\end{tikzcd}
$$

Thus there results a homomorphism connecting the 1st row to the 4th. Therefore we may extend the 3rd row beyond the broken arrow (a) as indicated in the brackets, and this induces the homomorphism Δ which leads to the long exact sequence

$$\cdots \to E_{n+1}(A', B) \to E_{n+1}(A, B) \to E_{n+1}(A'', B)$$
$$\xrightarrow{\Delta} E_n(A', B) \to E_n(A, B) \to E_n(A'', B) \to \cdots.$$

This long exact sequence terminates at $n=1$, namely

$$\cdots \xrightarrow{\Delta} E_1(A', B) \to E_1(A, B) \to E_1(A'', B).$$

Also, an exact sequence

$$0 \to T(A'_1, B) \to T(A_1, B) \to T(A''_1, B) \to E_1(A', B)$$
$$\to E_1(A, B) \to E_1(A'', B)$$

results.

A question that naturally arises is: Is there a natural way of defining $E_n(A, B)$ for $n<1$? In particular, how is T related to E_0?

Coming back to the T-dimension, we have the relations:

(i) $E_1(A, B)=0$ for all B in $\boldsymbol{B}$ iff $A \in \mathscr{P}(\boldsymbol{A})$.

(ii) $E_n(A, B)=0$ for all B in $\boldsymbol{B}$ iff T-dim $A \leqq n$.

One verifies easily the effect on the results upon changing the hypothesis on T, say being contravariant in one or both of the variables, or being exact on the right rather than on the left.

For instance, if T is right exact then the long exact sequence ends as expected, namely, $E_1(A'', B) \to T(A', B) \to T(A, B) \to T(A'', B) \to 0$. In particular, we may set $E_0=T$ and $E_k=0$ for $k<0$.

One easily derives the application to Ext and Tor. For Ext the equivalence reduces to the projective (injective) equivalence.

A natural question that arises is: If $\mathrm{Tor}_1(A, -)\simeq\mathrm{Tor}_1(B, -)$ are A and B flat equivalent (with respect to $\otimes$). (For the similar question concerning $\mathrm{Ext}^1(\ ,\ )$, see [**1**].) A partial answer is:

THEOREM 5. *Let R be a Dedekind domain, and let* $\mathrm{Tor}_1(A, -)\simeq \mathrm{Tor}_1(B, -)$. *Then A and B are flat equivalent.*

PROOF. Let Q be the quotient field of R. A straightforward reasoning yields the equalities $t(M)=\mathrm{Tor}_1(M, Q/R)$ for every module M, where $t(M)$ denotes the torsion submodule of M. Also, for a module M over a Dedekind domain R it is well known that M is a flat module iff M is a torsionless module. Therefore, in the situation under consideration we have (i) $t(A)\simeq t(B)$ and (ii) $A/t(A)$ and $B/t(B)$ are flat modules. The sequence A, $t(A)$, B thus yields the flat equivalence of A and B.

One can use the equivalence notion in order to get a deeper insight into duality homomorphism, and to obtain their proofs by induction on the iterations rather than by the method of bicomplexes and spectral sequences. Even though the difference does not seem to be crucial, some finer analysis may be achieved (see [**4**] for details).

For the rest, iterations and equivalences stand for projective iterations and projective equivalences.

Let R and S be rings, let A be a left R-module, let B be a left R- right S-bimodule, and let C be a right S-module. Then for every i, the homomorphism ψ_i, $\psi_i:\mathrm{Tor}_i^R(\mathrm{Hom}_S(B, C), A)\to\mathrm{Hom}_S(\mathrm{Ext}_R^i(A, B), C)$, is naturally defined. We obtain:

PROPOSITION 6. *If the equivalence class of A_i contains a finitely generated module, then ψ_i is an epimorphism, and if A_i is a projective module, then ψ_i is a monomorphism.*

Also, one can derive conditions under which ψ_i is an isomorphism. This can be used to deduce:

COROLLARY 7. *If the right injective envelope of R is a flat (right) module, and if the equivalence class of A_i contains a finitely generated module, then* $\mathrm{Hom}(\mathrm{Ext}^n(A, R), R)=0$.

COROLLARY 8. *Let A be a flat module. If the equivalence class of A_1 contains a finitely generated module, then A is a projective module.*

COROLLARY 9. *Let R be a left Noetherian ring and let every right injective module be a right flat module. Then R is a quasi-Frobenius ring, that is, right and left Artinian self-injective ring.*

We wish to conclude by pointing out a way to obtain a "Poincaré duality"-type theorem.

First observe that a homomorphism θ may be obtained from $\mathrm{Ext}^1(B, C)$ into $\mathrm{Ext}^1(B_1, C_1)$ by fixing projective modules P, P' via the diagram (commutative with exact rows and columns):

$$
\begin{array}{lccccccccc}
 & & & 0 & & 0 & & 0 & & \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
(\mathrm{E}') & & 0 & \to & C_1 & \longrightarrow & X_1 & \longrightarrow & B_1 & \to 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
(*) & & 0 & \to & P' & \to & P' \oplus P & \to & P & \longrightarrow 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
(\mathrm{E}) & & 0 & \to & C & \longrightarrow & X & \longrightarrow & B & \to 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & 0 & & 0 & & 0 & &
\end{array}
$$

that is: $\theta(E)=E'$.

Next, one considers the resulting diagram (commutative with exact rows and columns):

$$
\begin{array}{lccccccccc}
 & & & 0 & & 0 & & 0 & & \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & \mathrm{Tor}_1(A, C) & \to & \mathrm{Tor}_1(A, X) & \longrightarrow & \mathrm{Tor}_1(A, B) & & \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
 & \mathrm{Tor}_1(A, B_1) & \to & A \otimes C_1 & \longrightarrow & A \otimes X_1 & \longrightarrow & A \otimes B_1 & \to & 0 \\
(**) & & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & \to & A \otimes P_1 & \to & A \otimes (P' \oplus P) & \longrightarrow & A \otimes P & \longrightarrow & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & A \otimes C & \longrightarrow & A \otimes X & \longrightarrow & A \otimes B & \to & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & 0 & & 0 & & 0 & &
\end{array}
$$

which by the snake lemma yields a homomorphism $\mathrm{Tor}_1(A, B) \to A \otimes C$.

Following this last homomorphism on the diagram (**) and fixing an element in $\mathrm{Tor}_1(A, B)$ we obtain a homomorphism: $\mathrm{Ext}^1(B, C) \to A \otimes C$.

A careful analysis yields that the homomorphism thus obtained of $\mathrm{Tor}_1(A, B) \times \mathrm{Ext}^1(B, C) \to A \otimes C$ is bilinear, whence we conclude by a simple induction on iterations:

THEOREM 10. *For $i \geqq 0$ there results a homomorphism* $\mathrm{Tor}_i(A, B) \otimes_Z \mathrm{Ext}^i(B, C) \to A \otimes C$.

Checking the upper left corner of (**)

$$
\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
 & & \mathrm{Tor}_1(A, C) \\
 & & \downarrow \\
\mathrm{Tor}_1(A, B_1) & \to & A \otimes C_1
\end{array}
$$

and using the isomorphism of $\mathrm{Tor}_1(A, B_1)$ with $\mathrm{Tor}_2(A, B)$, we reach

THEOREM 11. *For* $i \geqq j \geqq 1$ *there results a homomorphism* $\mathrm{Tor}_i(A, B) \otimes_Z \mathrm{Ext}^j(B, C) \to \mathrm{Tor}_{i-j}(A, C)$.

Further properties of these homomorphisms as to their naturality can be checked.

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