THE NUMBER OF FIELD TOPOLOGIES ON COUNTABLE FIELDS

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Abstract. J. O. Kiltinen proves that every infinite field admits a nondiscrete, Hausdorff field topology. In this note it is shown that every countable field $K$ admits $2^{2^\omega}$ many field topologies, which even fail to be the join of locally bounded ring topologies.

1. Introduction. In §2 we give a method for generating a fundamental system $\{V_n|n \in \omega\}$ of neighborhoods of zero for a field topology on $K$. To do this, we first define the notion of a condition. This is a function from $\omega \times \{0, 1\}$ into the set of finite subsets of $K$ with some further properties. A condition $p$ decides for a finite number of elements of $K$ if they are elements of $V_n$ or not by saying $r$ is an element of $V_n$ if $r \in p(n, 0)$ and $r$ is not an element of $V_n$ if $r \not\in p(n, 1)$. Given two conditions $p$ and $p'$, then $p'$ extends $p$ if $p(n, i) \subseteq p'(n, i)$ for every $(n, i)$. If $G$ is a chain of conditions, then by the above decision process we get a fundamental system of a field topology.

Since a condition decides only for a finite number of elements of $K$ if they are elements of $V_n$ or not, we can prove in §3 that there are “many” possibilities to extend a condition. In §4 we define what it means for a set of chains of conditions to be entwined. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are different nonempty subsets of an entwined set then the join-topologies $\bigvee \{\mathcal{F}_G|G \in \mathcal{G}_1\}$ and $\bigvee \{\mathcal{F}_G|G \in \mathcal{G}_2\}$ are also different. Using the results of §3 we can construct an entwined set of power $2^{2^\omega}$ of chains of conditions in such a way that for every subset $\mathcal{G}_3$, the topology $\bigvee \{\mathcal{F}_G|G \in \mathcal{G}_3\}$ is not the join of locally bounded ring topologies.

2. Chains of conditions. Let $(K, +, \cdot, 0, 1)$ be a countable field and let $\phi_K$ denote the set of functions $\varphi_d, \varphi_c, \varphi_b$ and $\varphi_a, a \in K$, defined by $\varphi_d(X) = X/(1-X), \varphi_c(X) = X \cdot X, \varphi_b(X) = X - X$ and $\varphi_a(X) = a \cdot X$, for every subset $X$ of $K$ with $1 \notin X$. Since $K$ is countable there is a sequence $(\varphi_n)_{n \in \omega}$ of elements of $\phi_K$ such that the set $\{n|\varphi = \varphi_n\}$ is infinite for every $\varphi \in \phi_K$. A sequence $(V_n|n \in \omega)$ of subsets of $K$ is called a fundamental system, if $1 \notin V_n, V_{n+1} \subseteq V_n, \varphi_n(V_{n+1}) \subseteq V_n$. For every field topology there is a basic...
system of neighborhoods of zero which is a fundamental system, and every fundamental system determines a field topology.

1. Definition. A function $p$ from $\omega \times \{0, 1\}$ into the set of all finite subsets of $K$ is called a condition, if the following properties hold:

   (a) $0 \in p(n, 0)$ and $1 \in p(n, 1)$,
   (b) $p(n, 0) \cap p(n, 1) = \emptyset$,
   (c) $p(n+1, i) \subseteq p(n, i)$,
   (d) $\varphi_n(p(n+1, 0)) \subseteq p(n, 0)$.

Let $P$ be the set of all conditions. $P$ is not empty, since $p^0$ defined by $p(n, i) = \{i\}$ for every $n \in \omega$, is an element of $P$. If $p$ and $p'$ are two conditions, we say that $p'$ extends $p$ (written $p \leq p'$), if $p(n, i)$ is a subset of $p'(n, i)$ for every $(n, i) \in \omega \times \{0, 1\}$. $\leq$ is a partial ordering of $P$. If $G$ is a chain of conditions, then $V^G_n$ is defined to be $\bigcup \{p(n, 0) \mid p \in G\}$.

2. Theorem. Let $G$ be a chain of conditions, then $\{V^G_n \mid n \in \omega\}$ is a fundamental system.

   The proof is straightforward if we use the fact that
   $$\varphi_n(\bigcup \{p(n + 1, 0) \mid p \in G\}) = \bigcup \{\varphi_n(p(n + 1, 0)) \mid p \in G\}.$$

   For every chain of conditions let $T_G$ denote the field topology which is determined by $\{V^G_n \mid n \in \omega\}$.

3. Theorem. If $T$ is a field topology with a countable basis, then there is a chain $G$ of conditions such that $T = T_G$.

   Proof. Let $\{V^G_n \mid n \in \omega\}$ be a fundamental system which determines the topology $T$ and let $(r^n_k)_{k \in \omega}$ be a well ordering of $V^G_n$ for each $n \in \omega$. By recursion we define for every $m$ a condition $p_m$ as follows:

   $$p_m(n, i) = p_m(n + 1, i) \cup \varphi_n(p_m(n + 1, i)) \cup \{r^n_j \mid 0 \leq j \leq m - n\}$$
   if $n \leq m$ and $i = 0$,

   $$= \{i\} \text{ otherwise.}$$

   Let $G$ be the set $\{p_m \mid m \in \omega\}$. Then we have for every $n$ that $V^G_n = V^G_n$ and therefore is $T_G = T$. □

3. Extensions of conditions. Here we prove that for every condition $p$, for each $(n, i) \in \omega \times \{0, 1\}$ and for nearly all (this means that for all but finitely many) $r \in K$ there exists a condition $p_r \geq p$ such that $r \in p_r(n, i)$. This will enable us to prove in §3 that there are "many" different chains of conditions. First the easy case.

4. Theorem. Let $p$ be a condition and $n \in \omega$. Then for nearly all $r \in K$ there is a condition $p_r$ such that $p_r \geq p$ and $r \in p_r(n, 1)$.
Proof. Let \( r \in K, r \notin p(0, 0) \). If we define \( p_r \) as follows:

\[
p_r(n, i) = p(n, i) \cup \{r\} \quad \text{if } i = 1,
= p(n, i) \quad \text{otherwise},
\]

then \( p_r \) has the desired properties. Since \( p(0, 0) \) is finite, this holds for nearly all \( r \). \( \square \)

Now the other case. Let \( R(K) \) be the set of all rational functions over \( K \). If \( H \) is a subset of \( R(K) \) and if for each \( f \in H, r \in K \) is in the domain of \( f \), then \( H(r) \) shall denote the set of all \( f(r) \), with \( f \in H \). Let \( f \in R(K) \) and \( a \in K \) such that \( f(0) \neq a \). Then for nearly all \( r \in K, f(r) \neq a \). So we obtain:

5. Lemma. Let \( H \subseteq R(K) \) and \( M \subseteq K \) be finite. If \( H(0) \cap M = \emptyset \), then for nearly all \( r \in K, H(r) \cap M = \emptyset \).

6. Lemma. Let \( \varphi \in \phi_K \) and let \( H \) be a finite subset of \( R(K) \) with \( 1 \notin H(0) \), then there is a finite \( H' \subseteq R(K) \) such that, for nearly all \( r \in K, \varphi(H(r)) = H'(r) \).

Proof. If \( \varphi = \varphi_d \) we define \( H' \) to be \( \{fg(1-g) | f, g \in H \} \). Since \( 1 \notin H(0) \) we have for nearly all \( r \in K, 1 \notin H(r) \). Let \( r \) be such an element of \( K \). Then \( \varphi_d(H(r)) = H(r)/(1-H(r)) = H'(r) \). Thus, for nearly all \( r \in K, \varphi_d(H(r)) = H'(r) \). The proof is similar if \( \varphi = \varphi_c, \varphi_b \) or \( \varphi_a \). \( \square \)

Now let \( p \in P \) and let \( H \) be a finite subset of \( R(K) \). By induction over \( n \) we can prove

7. Theorem. If \( H(0) \) is a subset of \( p(n, 0) \), then for nearly all \( r \in K \) there is a condition \( p_r \geq p \) such that:

1. \( H(r) \subseteq p_r(n, 0) \),
2. \( p_r(m, i) = p(m, i) \) if \( m > n \) or \( i = 1 \).

(i) The Theorem holds for \( n = 0 \).

Proof. Since \( H(0) \) is a subset of \( p(n, 0) \), \( H(0) \cap p(0, 1) = \emptyset \). By Lemma 5, we have for nearly all \( r \in K \) that \( p(0, 1) \cap H(r) = \emptyset \). Let \( r \) be such an element. If \( p_r \) is defined by

\[
p_r(m, i) = p(m, i) \cup H(r) \quad \text{if } m = 0 \text{ and } i = 0,
= p(m, i) \quad \text{otherwise},
\]

then \( p_r \) has the desired properties.

(ii) Assume the Theorem holds for \( n \), then it holds for \( n + 1 \).

Proof. First we choose a finite subset \( H'' \) of \( R(K) \), with \( H''(0) \subseteq p(n, 0) \) as follows: Let \( H' = H \cup \{f_a | a \in p(n+1, 0)\} \), where \( f_a \) is the function defined by \( f_a(r) = a \) for every \( r \in K \). Since \( 1 \notin p(n+1, 0) \) and \( H'(0) \subseteq p(n+1, 0) \), we have by Lemma 6 that there is a finite \( H'' \subseteq R(K) \) such that
for nearly all \( r \in K \), \( H''(r) = q_n(H'(r)) \). Let \( L \) be the set of these \( r \)'s. If we define \( H''' = H' \cup H'' \), then we have that \( H'''(0) \subseteq p(n, 0) \). By assumption there are for nearly all \( r \in K \) conditions \( p'_r \supseteq p \) such that:

1. \( H'''(r) \subseteq p'_r(n, 0) \),
2. \( p(m, i) = p'_r(m, i) \) if \( m > n \) or \( i = 1 \).

Let \( L' \) be the set of these \( r \)'s and let \( r \in L \cap L' \). Then we define \( p_r \) by

\[
p_r(m, i) = p'_r(m, i) \cup H(r) \quad \text{if } m = n + 1 \text{ and } i = 0,
\]

\[
= p'_r(m, i) \quad \text{otherwise}.
\]

\( p_r \) has the desired properties. Since nearly all \( r \in K \) are in \( L \cap L' \), the theorem holds for \( n + 1 \).

8. Corollary. Let \( p \) be a condition and \( n \in \omega \). Then for nearly all \( r \in K \) there are conditions \( p_r \supseteq p \), such that \( r \in p_r(n, 0) \).

Proof. Take \( H \) to be \( \{\text{id}\} \), where \( \text{id} \) is the function which maps every element of \( K \) onto itself. By Theorem 7 we get the desired result.  \( \square \)

4. Entwined sets of chains of conditions. To prove that there are \( 2^{2^\aleph_0} \) many field topologies on \( K \), it suffices to show that there is a set \( \mathcal{G} \) of power \( 2^{\aleph_0} \) of chains of conditions such that for any two different non-empty subsets \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) of \( \mathcal{G} \) the join-topologies \( \bigvee \{\mathcal{F}_G|G \in \mathcal{G}_1\} \) and \( \bigvee \{\mathcal{F}_G|G \in \mathcal{G}_2\} \) are also different.

9. Definition. Let \( \mathcal{G} \) be a set of chains of conditions. \( \mathcal{G} \) is called entwined if, for every \( n \in \omega \) and for every finite subset \( \{G_i|0 \leq i \leq m\} \) of \( \mathcal{G} \), there are conditions \( p_i \in G_i \), \( 0 \leq i \leq m \), such that

\[
\bigcap \{p_i(n, 0) \mid 1 \leq i \leq m\} \cap p_0(0, 1) \quad \text{is not empty}.
\]

An easy consequence of Definition 9 is that there is a sequence which converges to zero in all of the topologies \( \mathcal{T}_{a_1}, \ldots, \mathcal{T}_{a_m} \) but which is bounded away from zero in \( \mathcal{T}_{a_0} \).

10. Theorem. If \( \mathcal{G} \) is entwined, then \( \bigvee \{\mathcal{F}_G|G \in \mathcal{G}\}\{G_0\} \) is not finer than \( \mathcal{T}_{a_0} \) for every \( G_0 \in \mathcal{G} \).

Proof. Suppose \( \bigvee \{\mathcal{F}_G|G \in \mathcal{G}\}\{G_0\} \) is finer than \( \mathcal{T}_{a_0} \) for some \( G_0 \in \mathcal{G} \). Then there are \( G_1, \ldots, G_m \in \mathcal{G} \) and \( k_1, \ldots, k_m \in \omega \) such that

\[
\bigcap \{V_{G_i}^a \mid 1 \leq i \leq m\} \subset V_{G_0}^a.
\]

Let \( k_0 = \max\{k_i|1 \leq i \leq m\} \). Since \( \mathcal{G} \) is entwined, there are conditions \( p_i \in G_i \), \( 0 \leq i \leq m \), such that \( M = \bigcap \{p_i(k_0, 0) \mid 1 \leq i \leq m\} \cap p_0(0, 1) \) is not empty. If \( r \in M \), then \( r \in V_{k_0}^{G_i} \subset V_{k_1}^{G_i} \) and \( r \in V_{k_0}^{G_0} \). This is a contradiction.  \( \square \)
11. Corollary. Let $\mathcal{G}$ be entwined and $\mathcal{G}_1, \mathcal{G}_2$ nonempty different subsets of $\mathcal{G}$. Then $\bigvee \{\mathcal{T}_G|G \in \mathcal{G}_1\}$ and $\bigvee \{\mathcal{T}_G|G \in \mathcal{G}_2\}$ are different.

Proof. Let $\mathcal{G}_1, \mathcal{G}_2 \subset \mathcal{G}$ such that $\mathcal{G}_1 \neq \mathcal{G}_2$. We may suppose that there is a $G_0 \in \mathcal{G}_1$ with $G_0 \notin \mathcal{G}_2$. Because $\mathcal{G}$ is entwined $\bigvee \{\mathcal{T}_G|G \in \mathcal{G}\}$ is not finer than $\mathcal{T}_{G_0}$. $\bigvee \{\mathcal{T}_G|G \in \mathcal{G}\}$ is finer than $\bigvee \{\mathcal{T}_G|G \in \mathcal{G}_2\}$ and $\bigvee \{\mathcal{T}_G|G \in \mathcal{G}_1\}$ is finer than $\mathcal{T}_{G_0}$. Thus, $\bigvee \{\mathcal{T}_G|G \in \mathcal{G}_1\}$ is not finer than $\bigvee \{\mathcal{T}_G|G \in \mathcal{G}_2\}$. □

Now we want to show that there is an entwined $\mathcal{G}$ of power $2^{\aleph_0}$. We identify each natural number with the set of its predecessors. $n^2$ denotes the set of functions from $n$ into $2$ and $\omega_2$ the set of those from $\omega$ into $2$. If $f$ is such a function, then $f|n$ is the restriction of $f$ to $n$. By induction over $n$, we shall choose, for each $f \in n^{n+2}$, a condition $p_f$ such that:

1. $p_f|n \leq p_f$.
2. $p_f'(0, 1) \cap \{\{p^g(n, 0)|g \in n^{n+2} \text{ and } g \neq f\}$ is not empty.
3. There is an $r \in n \cap \{\{p^g(n, 0)|g \in n^{n+2}\}$ and an $m \in \omega$, $m \neq 0$, such that $r^m \in n \cap \{p^g(0, 1)|g \in n^{n+2}\}$.

Let $p^f = p^0$. Assume that, for $f \in n^2$, $p_f'$ is chosen. Let $(f_k)_{k \in n^m}$ be a well ordering of $n^{n+2}$. For each $f \in n^{n+2}$ we choose, by induction over $k \in m_n+1$ conditions, $p'_k$ as follows: By Corollary 8 there are conditions $p'_k$ such that $p'_k \geq p^f|n$ and $M = \bigcap \{p'_k(n, 0)|f \in n^{n+2}\} \setminus \{0\} \neq \emptyset$. Let $r \in M$ be given. If there is an $m \in \omega$ such that $r^m = 1$, then take $p'_k$ to be $p_k'$. If there is no $m \in \omega$ such that $r^m = 1$, then $\{r^m|m \in \omega\}$ is infinite. Hence, by Theorem 4 there are conditions $p'_0 \geq p'_k$ and an $m \in \omega$ such that $r^m \in n \cap \{p^g(0, 1)|f \in n^{n+2}\}$.

Suppose $p'_k$ is already chosen. Then by Theorem 4 and Corollary 8 there are conditions $p'_{k+1}$ such that $p'_{k+1} \geq p'_k$ and $p'_{k+1}(0, 1) \cap \{p'_{k+1}(n, 0)|f \in n^{n+2} \text{ and } f \neq f'_k\}$ is nonempty. Let $p'_k = p'^{n+2}_m$. Then the conditions $p'_f$, $f \in n^{n+2}$, have the desired properties. Now for $g \in \omega^2$, define $G_g$ to be the chain $\{p'^{|n}n|n \in \omega\}$, and define $\mathcal{G}$ to be $\{G_g|g \in \omega^2\}$.

12. Theorem. $\mathcal{G}$ is an entwined set of power $2^{\aleph_0}$ of chains of conditions.

Proof. It is sufficient to show that $\mathcal{G}$ is entwined. Let $g_0 \in \omega_2$ and let $g_i \in \omega^2 \setminus \{g_0\}$, $1 \leq i \leq m$. Then for each $n$ there is a $k>n$ such that $g_0|k \notin \{g_i|k \leq i \leq m\}$. By 2, we know that $p^{|k}_0(0, 1) \cap \{p^{|k}_0(k-1, 0)|1 \leq i \leq m\}$ is not empty. Since $k-1 \leq n$ we have that $p^{|k}_0(k-1, 0)$ is a subset of $p^{|k}_0(n, 0)$. This implies that $p^{|k}_0(0, 1) \cap \{p^{|k}_0(n, 0)|1 \leq i \leq m\}$ is not empty. Thus, $\mathcal{G}$ is entwined. □

Now we shall prove that we have constructed $\mathcal{G}$ in such a way that for each $\mathcal{G}_0 \subset \mathcal{G}$ the topology $\bigvee \{\mathcal{T}_G|G \in \mathcal{G}_0\}$ is not the join of locally bounded ring topologies. By [1] it is sufficient to show that there is a neighborhood $V$ of zero such that for every neighborhood $U \subset V$ there is an $n \in \omega$ with $U^n \subset V$.
13. THEOREM. For every $\mathcal{G}' \subseteq \mathcal{G}$, $\bigvee \{ T_G | G \in \mathcal{G}' \}$ fails to be the join of locally bounded ring topologies.

PROOF. Let $G_0 \in \mathcal{G}'$ be given and let $U$ be a neighborhood of zero, $U \subseteq V_{G_0}^{G_0}$. Then there are finitely many $G_i \in \mathcal{G}'$, $1 \leq i \leq m$, and a $k \in \omega$ such that $\bigcap \{ V_{G_i}^{G_i} | 1 \leq i \leq m \} \subseteq U$. By the definition of $\mathcal{G}$ there are functions $f_i \in k^{+1.2}$, $0 \leq i \leq m$, such that $p^{f_i}(k, 0) \subseteq V_{G_i}^{G_i}$. From 3 it follows that there is an $r \in \bigcap \{ p^{f_i}(k, 0) | 1 \leq i \leq m \}$ and a $z \in \omega$, $z \neq 0$, such that $r^z \in p^{f_i}(0, 1)$. Hence, $U^z \subseteq V_{G_0}^{G_0}$ and therefore $\bigvee \{ T_G | G \in \mathcal{G}' \}$ is not the join of locally bounded ring topologies.

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