THE NUMBER OF FIELD TOPOLOGIES
ON COUNTABLE FIELDS

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Abstract. J. O. Kiltinen proves that every infinite field admits a nondiscrete, Hausdorff field topology. In this note it is shown that every countable field $K$ admits $2^{|\mathbb{N}|}$ many field topologies, which even fail to be the join of locally bounded ring topologies.

1. Introduction. In §2 we give a method for generating a fundamental system $\{V_n|n \in \omega\}$ of neighborhoods of zero for a field topology on $K$. To do this, we first define the notion of a condition. This is a function from $\omega \times \{0, 1\}$ into the set of finite subsets of $K$ with some further properties. A condition $p$ decides for a finite number of elements of $K$ if they are elements of $V_n$ or not by saying $r$ is an element of $V_n$ if $r \in p(n, 0)$ and $r$ is not an element of $V_n$ if $r \notin p(n, 1)$. Given two conditions $p$ and $p'$, then $p'$ extends $p$ if $p(n, i) \subseteq p'(n, i)$ for every $(n, i)$. If $G$ is a chain of conditions, then by the above decision process we get a fundamental system of a field topology.

Since a condition decides only for a finite number of elements of $K$ if they are elements of $V_n$ or not, we can prove in §3 that there are "many" possibilities to extend a condition. In §4 we define what it means for a set of chains of conditions to be entwined. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are different non-empty subsets of an entwined set then the join-topologies $\bigvee \{V_g|G \in \mathcal{G}_1\}$ and $\bigvee \{V_g|G \in \mathcal{G}_2\}$ are also different. Using the results of §3 we can construct an entwined set of power $2^{|\mathbb{N}|}$ of chains of conditions in such a way that for every subset $\mathcal{G}_1$, the topology $\bigvee \{V_g|G \in \mathcal{G}_1\}$ is not the join of locally bounded ring topologies.

2. Chains of conditions. Let $(K, +, \cdot, 0, 1)$ be a countable field and let $\phi_K$ denote the set of functions $\varphi_d$, $\varphi_c$, $\varphi_b$, and $\varphi_a$, $a \in K$, defined by $\varphi_d(X) = X/(1-X)$, $\varphi_c(X) = X \cdot X$, $\varphi_b(X) = X - X$ and $\varphi_a(X) = a \cdot X$, for every subset $X$ of $K$ with $1 \notin X$. Since $K$ is countable there is a sequence $(\varphi_n)_{n \in \omega}$ of elements of $\phi_K$ such that the set $\{n|\varphi = \varphi_n\}$ is infinite for every $\varphi \in \phi_K$. A sequence $\{V_n|n \in \omega\}$ of subsets of $K$ is called a fundamental system, if $1 \notin V_n$, $V_{n+1} \subseteq V_n$, $\varphi_n(V_{n+1}) \subseteq V_n$. For every field topology there is a basic
system of neighborhoods of zero which is a fundamental system, and every 
fundamental system determines a field topology.

1. Definition. A function \( p \) from \( \omega \times \{0, 1\} \) into the set of all finite 
subsets of \( K \) is called a condition, if the following properties hold:
(a) \( 0 \in p(n, 0) \) and \( 1 \in p(n, 1) \),
(b) \( p(n, 0) \cap p(n, 1) = \emptyset \),
(c) \( p(n+1, i) \subseteq p(n, i) \),
(d) \( \varphi_n(p(n+1, 0)) \subseteq p(n, 0) \).

Let \( P \) be the set of all conditions. \( P \) is not empty, since \( p^0 \) defined by
\( p(n, i) = \{i\} \) for every \( n \in \omega \), is an element of \( P \). If \( p \) and \( p' \) are two condi-
tions we say that \( p' \) extends \( p \) (written \( p \leq p' \)), if \( p(n, i) \) is a subset of
\( p'(n, i) \) for every \( (n, i) \in \omega \times \{0, 1\} \). \( \leq \) is a partial ordering of \( P \). If \( G \) is a
chain of conditions, then \( V_n^G \) is defined to be \( \bigcup \{ p(n, 0) \mid p \in G \} \).

2. Theorem. Let \( G \) be a chain of conditions, then \( \{ V_n^G \mid n \in \omega \} \) is a 
fundamental system.

The proof is straightforward if we use the fact that
\[ \varphi_n(\bigcup \{ p(n+1, 0) \mid p \in G \}) = \bigcup \{ \varphi_n(p(n+1, 0)) \mid p \in G \} \]

For every chain of conditions let \( T_G \) denote the field topology which is
determined by \( \{ V_n^G \mid n \in \omega \} \).

3. Theorem. If \( T \) is a field topology with a countable basis, then there
is a chain \( G \) of conditions such that \( T = T_G \).

Proof. Let \( \{ V_n \mid n \in \omega \} \) be a fundamental system which determines the
topology \( T \) and let \( (r^n_k)_{k \in \omega} \) be a well ordering of \( V_n \) for each \( n \in \omega \). By
recursion we define for every \( m \) a condition \( p_m \) as follows:
\[ p_m(n, i) = p_m(n+1, i) \cup \varphi_n(p_m(n+1, i)) \cup \{ r^n_j \mid 0 \leq j \leq m - n \} \]
if \( n \leq m \) and \( i = 0 \),
\[ = \{i\} \text{ otherwise.} \]

Let \( G \) be the set \( \{ p_m \mid m \in \omega \} \). Then we have for every \( n \) that \( V_n^G = V_n \) and
therefore is \( T_G = T \). \( \square \)

3. Extensions of conditions. Here we prove that for every condition \( p \),
for each \( (n, i) \in \omega \times \{0, 1\} \) and for nearly all (this means that for all but
finitely many) \( r \in K \) there exists a condition \( p_r \leq p \) such that \( r \in p_r(n, i) \).
This will enable us to prove in §3 that there are "many" different chains of
conditions. First the easy case.

4. Theorem. Let \( p \) be a condition and \( n \in \omega \). Then for nearly all
\( r \in K \) there is a condition \( p_r \) such that \( p_r \leq p \) and \( r \in p_r(n, 1) \).
Proof. Let \( r \in K, r \notin p(0, 0) \). If we define \( p_r \) as follows:

\[
p_r(n, i) = \begin{cases} 
p(n, i) \cup \{r\} & \text{if } i = 1, \\
p(n, i) & \text{otherwise}, 
\end{cases}
\]

then \( p_r \) has the desired properties. Since \( p(0, 0) \) is finite, this holds for nearly all \( r \). □

Now the other case. Let \( R(K) \) be the set of all rational functions over \( K \). If \( H \) is a subset of \( R(K) \) and if for each \( f \in H, r \in K \) is in the domain of \( f \), then \( H(r) \) shall denote the set of all \( f(r) \), with \( f \in H \). Let \( f \in R(K) \) and \( a \in K \) such that \( f(0) \neq a \). Then for nearly all \( r \in K, f(r) \neq a \). So we obtain:

5. Lemma. Let \( H \subseteq R(K) \) and \( M \subseteq K \) be finite. If \( H(0) \cap M = \emptyset \), then for nearly all \( r \in K, H(r) \cap M = \emptyset \).

6. Lemma. Let \( \varphi \in \Phi_K \) and let \( H \) be a finite subset of \( R(K) \) with \( 1 \notin H(0) \), then there is a finite \( H' \subseteq R(K) \) such that, for nearly all \( r \in K, \varphi(H(r)) = H'(r) \).

Proof. If \( \varphi = \varphi_d \) we define \( H' \) to be \( \{ f(1-g) | f, g \in H \} \). Since \( 1 \notin H(0) \) we have for nearly all \( r \in K, 1 \notin H(r) \). Let \( r \) be such an element of \( K \). Then \( \varphi_d(H(r)) = H(r)/(1-H(r)) = H'(r) \). Thus, for nearly all \( r \in K, \varphi_d(H(r)) = H'(r) \). The proof is similar if \( \varphi = \varphi_e, \varphi_b \) or \( \varphi_a \). □

Now let \( p \in P \) and let \( H \) be a finite subset of \( R(K) \). By induction over \( n \) we can prove

7. Theorem. If \( H(0) \) is a subset of \( p(n, 0) \), then for nearly all \( r \in K \) there is a condition \( p_r \geq p \) such that:

1. \( H(r) \subseteq p_r(n, 0) \),
2. \( p_r(m, i) = p(m, i) \) if \( m > n \) or \( i = 1 \).

(i) The Theorem holds for \( n = 0 \).

Proof. Since \( H(0) \) is a subset of \( p(n, 0), H(0) \cap p(0, 1) = \emptyset \). By Lemma 5, we have for nearly all \( r \in K \) that \( p(0, 1) \cap H(r) = \emptyset \). Let \( r \) be such an element. If \( p_r \) is defined by

\[
p_r(m, i) = \begin{cases} 
p(m, i) \cup H(r) & \text{if } m = 0 \text{ and } i = 0, \\
p(m, i) & \text{otherwise}, 
\end{cases}
\]

then \( p_r \) has the desired properties.

(ii) Assume the Theorem holds for \( n \), then it holds for \( n + 1 \).

Proof. First we choose a finite subset \( H'' \) of \( R(K) \), with \( H''(0) \subseteq p(n, 0) \) as follows: Let \( H' = H \cup \{ f_a(a \in p(n+1, 0)) \} \), where \( f_a \) is the function defined by \( f_a(r) = a \) for every \( r \in K \). Since \( 1 \notin p(n+1, 0) \) and \( H'(0) \subseteq p(n+1, 0) \), we have by Lemma 6 that there is a finite \( H'' \subseteq R(K) \) such that
for nearly all \( r \in K \), \( H'(r) = \varphi_n(H'(r)) \). Let \( L \) be the set of these \( r \)'s. If we define \( H'' = H' \cup H'' \), then we have that \( H''(0) \subseteq p(n, 0) \). By assumption there are for nearly all \( r \in K \) conditions \( p_r' \geq p \) such that:

1. \( H''(r) \subseteq p_r'(n, 0) \),
2. \( p(m, i) = p_r'(m, i) \) if \( m > n \) or \( i = 1 \).

Let \( L' \) be the set of these \( r \)'s and let \( r \in L \cap L' \). Then we define \( p_r \) by

\[
p_r(m, i) = p_r'(m, i) \cup H(r) \quad \text{if } m = n + 1 \quad \text{and} \quad i = 0,
\]

\[
= p_r'(m, i) \quad \text{otherwise}.
\]

\( p_r \) has the desired properties. Since nearly all \( r \in K \) are in \( L \cap L' \), the theorem holds for \( n + 1 \).

8. COROLLARY. Let \( p \) be a condition and \( n \in \omega \). Then for nearly all \( r \in K \) there are conditions \( p_r' \geq p \), such that \( r \in p_r(n, 0) \).

PROOF. Take \( H \) to be \( \{\text{id}\} \), where \( \text{id} \) is the function which maps every element of \( K \) onto itself. By Theorem 7 we get the desired result. \( \square \)

4. Entwined sets of chains of conditions. To prove that there are \( 2^{2^\aleph_0} \) many field topologies on \( K \), it suffices to show that there is a set \( \mathcal{G} \) of power \( 2^{2^\aleph_0} \) of chains of conditions such that for any two different non-empty subsets \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) of \( \mathcal{G} \) the join-topologies \( \bigvee \{\mathcal{F}_G | G \in \mathcal{G}_1\} \) and \( \bigvee \{\mathcal{F}_G | G \in \mathcal{G}_2\} \) are also different.

9. DEFINITION. Let \( \mathcal{G} \) be a set of chains of conditions. \( \mathcal{G} \) is called entwined if, for every \( n \in \omega \) and for every finite subset \( \{G_i | 0 \leq i \leq m\} \) of \( \mathcal{G} \), there are conditions \( p_i \in G_i \), \( 0 \leq i \leq m \), such that

\[
\bigcap \{p_i(n, 0) | 1 \leq i \leq m\} \cap p_0(0, 1) \quad \text{is not empty}.
\]

An easy consequence of Definition 9 is that there is a sequence which converges to zero in all of the topologies \( \mathcal{T}_{G_1}, \ldots, \mathcal{T}_{G_m} \) but which is bounded away from zero in \( \mathcal{T}_{G_0} \).

10. THEOREM. If \( \mathcal{G} \) is entwined, then \( \bigvee \{\mathcal{T}_G | G \in \mathcal{G} \setminus \{G_0\}\} \) is not finer than \( \mathcal{T}_{G_0} \) for every \( G_0 \in \mathcal{G} \).

PROOF. Suppose \( \bigvee \{\mathcal{T}_G | G \in \mathcal{G} \setminus \{G_0\}\} \) is finer than \( \mathcal{T}_{G_0} \) for some \( G_0 \in \mathcal{G} \). Then there are \( G_1, \ldots, G_m \in \mathcal{G} \) and \( k_1, \ldots, k_m \in \omega \) such that

\[
\bigcap \{V_{G_i}^{G_{k_i}} | 1 \leq i \leq m\} \subset V_0^{G_0}.
\]

Let \( k_0 = \max\{|k_i| | 1 \leq i \leq m\} \). Since \( \mathcal{G} \) is entwined, there are conditions \( p_i \in G_i \), \( 0 \leq i \leq m \), such that \( M = \bigcap \{p_i(k_0, 0) | 1 \leq i \leq m\} \cap p_0(0, 1) \) is not empty. If \( r \in M \), then \( r \in V_{k_0}^{G_i} \subset V_{k_i}^{G_{k_i}} \) and \( r \in V_0^{G_0} \). This is a contradiction. \( \square \)
11. Corollary. Let $\mathcal{G}$ be entwined and $\mathcal{G}_1, \mathcal{G}_2$ nonempty different subsets of $\mathcal{G}$. Then $\bigvee \{ \mathcal{F}_G | G \in \mathcal{G}_1 \} \neq \bigvee \{ \mathcal{F}_G | G \in \mathcal{G}_2 \}$.

Proof. Let $\mathcal{G}_1, \mathcal{G}_2 \subseteq \mathcal{G}$ such that $\mathcal{G}_1 \neq \mathcal{G}_2$. We may suppose that there is a $G_0 \in \mathcal{G}_1$ with $G_0 \notin \mathcal{G}_2$. Because $\mathcal{G}$ is entwined $\bigvee \{ \mathcal{F}_G | G \in \mathcal{G} \}\{\{G_0\}\}$ is not finer than $\mathcal{F}_{G_0}$. $\bigvee \{ \mathcal{F}_G | G \in \mathcal{G}_1 \} \{G_0\}$ is finer than $\bigvee \{ \mathcal{F}_G | G \in \mathcal{G}_2 \}$ and $\bigvee \{ \mathcal{F}_G | G \in \mathcal{G}_1 \}$ is finer than $\mathcal{F}_{G_0}$. Thus, $\bigvee \{ \mathcal{F}_G | G \in \mathcal{G}_2 \}$ is not finer than $\bigvee \{ \mathcal{F}_G | G \in \mathcal{G}_1 \}$.

Now we want to show that there is an entwined $\mathcal{G}$ of power $2^{n_2}$. We identify each natural number with the set of its predecessors. $n_2$ denotes the set of functions from $n$ into 2 and $n_2$ the set of those from $\omega$ into 2. If $f$ is such a function, then $f|n$ is the restriction of $f$ to $n$. By induction over $n$, we shall choose, for each $f \in n_2+1$, a condition $p'$ such that:

1. $p'|n \subseteq p'$.
2. $p'(0, 1) \cap \{ p^g(n, 0) | g \in n_2+1 \}$ is not empty.
3. There is an $r \in (n \cap \{ p^g(n, 0) | g \in n_2+1 \})$ and an $m \in \omega$, $m \neq 0$, such that $r^m \in \{ p^g(0, 1) | g \in n_2+1 \}$.

Let $p^* = p^0$. Assume that, for $f \in n_2$, $p'$ is chosen. Let $(f_k)_{k \in \omega}$ be a well ordering of $n_2+1$. For every $f \in n_2+1$ we choose, by induction over $k \in m_n+1$ conditions, $p_k'$ as follows: By Corollary 8 there are conditions $p_k'$ such that $p^{f_k}_k \supseteq p^0_k$ and $M = \bigcap \{ p^f_k(n, 0) | f \in n_2+1 \}\{0\} \neq \varnothing$. Let $r \in M$ be given. If there is an $m \in \omega$ such that $r^m = 1$, then take $p_k'$ to be $p^f_k$. If there is no $m \in \omega$ such that $r^m = 1$, then $\{ r^m | m \in \omega \}$ is infinite. Hence, by Theorem 4 there are conditions $p_0^g \supseteq p^0_k$ and an $m \in \omega$ such that $r^m \in \bigcap \{ p^g_k(0, 1) | f \in n_2+1 \}$. Suppose $k_k'$ is already chosen. Then by Theorem 4 and Corollary 8 there are conditions $p^f_{k+1}$ such that $p^f_{k+1} \supseteq p^f_k$ and $p^f_{k+1}(0, 1) \cap \{ p^g_{k+1}(n, 0) | f \in n_2+1 \}$ and $f \neq f_k'$ is nonempty. Let $p^* = p^m_{k+1}$. Then the conditions $p'$, $f \in n_2+1$, have the desired properties. Now for $g \in \omega_2$, define $G_g$ to be the chain $\{ p^{g_k}_k | n \in \omega \}$, and define $\mathcal{G}$ to be $\{ G_g | g \in \omega_2 \}$.

12. Theorem. $\mathcal{G}$ is an entwined set of power $2^{n_2}$ of chains of conditions.

Proof. It is sufficient to show that $\mathcal{G}$ is entwined. Let $g_0 \in \omega_2$ and let $g_i \in \omega_2\{ g_0 \}$, $1 \leq i \leq m$. Then for each $n$ there is a $k > n$ such that $g_k | k \notin \{ g_i | k \leq i \leq m \}$. By 2, we know that $p^{g_k}_k(0, 1) \cap \{ p^{g_k}_k(n, 0) | 1 \leq i \leq m \}$ is not empty. Since $k - 1 \geq n$ we have that $p^{g_k}_k(k+1, 0)$ is a subset of $p^{g_k}_k(n, 0)$. This implies that $p^{g_k}_k(0, 1) \cap \{ p^{g_k}_k(n, 0) | 1 \leq i \leq m \}$ is not empty. Thus, $\mathcal{G}$ is entwined.

Now we shall prove that we have constructed $\mathcal{G}$ in such a way that for each $\mathcal{G}_1 \subseteq \mathcal{G}$ the topology $\bigvee \{ \mathcal{F}_G | G \in \mathcal{G}_1 \}$ is not the join of locally bounded ring topologies. By [1] it is sufficient to show that there is a neighborhood $V$ of zero such that for every neighborhood $U \subseteq V$ there is an $n \in \omega$ with $U^n \subseteq V$.  

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13. Theorem. For every $\mathcal{G}' \subseteq \mathcal{G}$, $\bigvee \{ \mathcal{F}_{G} | G \in \mathcal{G}' \}$ fails to be the join of locally bounded ring topologies.

Proof. Let $G_0 \in \mathcal{G}'$ be given and let $U$ be a neighborhood of zero, $U \subseteq V_{0}^{G_0}$. Then there are finitely many $G_i \in \mathcal{G}'$, $1 \leq i \leq m$, and a $k \in \omega$ such that $\bigcap \{ V_{k}^{G_i} | 1 \leq i \leq m \} \subseteq U$. By the definition of $\mathcal{G}$ there are functions $f_i \in k^{+1}2$, $0 \leq i \leq m$, such that $p^{f_i}(k, 0) \subseteq V_{k}^{G_i}$. From 3 it follows that there is an $r \in \bigcap \{ p^{f_i}(k, 0) | 1 \leq i \leq m \}$ and a $z \in \omega$, $z \neq 0$, such that $r^z \in p^{f_0}(0, 1)$. Hence, $U^z \subseteq V_{0}^{G_0}$ and therefore $\bigvee \{ \mathcal{F}_{G} | G \in \mathcal{G}' \}$ is not the join of locally bounded ring topologies.

References


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