

THE LARGEST SUM-FREE SUBSEQUENCE FROM A SEQUENCE OF n NUMBERS

S. L. G. CHOI

ABSTRACT. Let $g(n)$ denote the largest integer so that from any sequence of n real numbers one can always select a sum-free subsequence of $g(n)$ numbers. Erdős has shown that $g(n) > 2^{-1/2}n^{1/2}$. In this paper we obtain an improved estimate by a different method.

1. For any natural number n , let $g(n)$ denote the largest integer so that from any sequence of n real numbers one can always select a sum-free subsequence of $g(n)$ numbers, namely, a subsequence of $g(n)$ numbers none of which is the sum of other numbers of the subsequence. An estimate of the type

$$(1) \quad g(n) > cn^{1/2}, \quad n \geq n_0$$

was first obtained by Erdős [1] with $c=2^{-1/2}$ and $n_0=1$. The purpose of this note is to establish (1) with $c=36/35$ and n_0 sufficiently large by an entirely different method. We mention that a slight improvement over $36/35$ is certainly possible by our method but it appears that a new idea may be needed to improve the constant to 2.¹

First, for the sake of comparison with our method, we recall Erdős' simple and elegant proof of (1) with $c=2^{-1/2}$.

Let a_1, \dots, a_n be a sequence of n numbers and T be large. Denote by \mathcal{J}_j the set in α , $0 < \alpha < T$ for which $a_j\alpha \pmod{1}$ is between $(2n)^{-1/2}$ and $2^{1/2}n^{-1/2}$, and by $m(\mathcal{J}_j)$ the measure of \mathcal{J}_j . Then

$$\left| m(\mathcal{J}_j) - \frac{T}{(2n)^{1/2}} \right| < A$$

where A depends at most on the a 's. From this inequality it follows that there is an α and at least $(n/2)^{1/2}$ a 's, say a_{i_j} ($1 \leq j \leq (n/2)^{1/2}$) such that $a_{i_j}\alpha \pmod{1}$ is between $(2n)^{-1/2}$ and $2^{1/2}n^{-1/2}$.

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¹ It has recently been brought to my knowledge that Professor Cantor has (essentially) improved the constant to $\sqrt{2}$ by a refinement of our method.

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2. A very simple argument suffices to prove (1) with $c=1$. Let

$$(2) \quad a_1 \geq a_2 \geq \dots \geq a_n$$

be a sequence of n real numbers. We select from the sequence (2) a set of k distinct numbers b_1, b_2, \dots, b_k satisfying

$$(3) \quad b_i \geq 2b_{i+1} \quad (i = 1, 2, \dots, k - 1)$$

as follows. Let $b_1 = a_1$ and suppose b_i ($i \geq 1$) has been chosen. Then we choose b_{i+1} to be the largest number from (2) not exceeding $\frac{1}{2}b_i$. Suppose b_k is the last number that can be so selected. The sequence $b_1 > b_2 > \dots > b_k$ is clearly sum-free and we have (1) with $c=1$ if $k \geq n^{1/2}$. Suppose now $k < n^{1/2}$. Then on putting $b_{k+1} = a_n - 1$, we see that there exists some j^* ($1 \leq j^* \leq k$) such that at least $n^{1/2}$ numbers of (2) lie in the interval $(b_{j^*+1}, b_{j^*}]$. These numbers are sum-free since $b_{j^*} \leq 2b_{j^*+1} + 1$.

3. We proceed to obtain (1) with $c=36/35$. We let the numbers b_1, \dots, b_k be defined as in §2. We may clearly assume

$$(4) \quad k \leq (36/35)n^{1/2} \quad \text{and}$$

$$(5) \quad |\mathcal{S}_j| \leq (36/35)n^{1/2} \quad (j = 1, \dots, k)$$

where \mathcal{S}_j denotes the set consisting of all numbers (not necessarily distinct) of (2) in the interval $(b_{j+1}, b_j]$, and where $|\mathcal{S}_j|$ denotes the number of elements of \mathcal{S}_j .

Let \mathcal{S}_{i_j} ($j=1, \dots, h$) denote those sets containing at least $(31/35)n^{1/2}$ elements. We have, on using (5)

$$(k - h)(31/35)n^{1/2} + h(36/35)n^{1/2} \geq n, \quad \text{i.e.} \quad h \geq 7(n^{1/2} - (31/35)k)$$

which gives, because of (4)

$$(6) \quad h \geq 7 \left(1 - \frac{31.36}{35^2} \right) n^{1/2}.$$

Let h_1 be defined by

$$(7) \quad h - h_1 - 1 > \frac{\log((36/35)n^{1/2})}{\log 2} \geq h - h_1 - 2.$$

We remark for future reference that a consequence of (7) is that every number in $\bigcup_{j=1}^{h_1} \mathcal{S}_{i_j}$ is larger than β which denotes the sum of all the numbers in \mathcal{S}_{i_j} . We also note that in view of (6) and (7) we have

$$(8) \quad h_1 \geq (3/5)n^{1/2}$$

if n is sufficiently large.

We now consider two cases.

Case 1. There exists some j^* ($1 \leq j^* \leq h_1$) such that $\mathcal{S}_{i_{j^*}}$ contains a set \mathcal{U} of $\geq |\mathcal{S}_{i_{j^*}}|/6$ distinct numbers differing from one another by more than β .

Case 2. For each $j=1, 2, \dots, h_1$, there exists in \mathcal{S}_{i_j} a set \mathcal{J}_j of 7 numbers all contained in an interval of length β .

If Case 1 holds then the set consisting of all the numbers in \mathcal{U} and \mathcal{S}_{i_h} is sum-free. To see this we first note that \mathcal{U} and \mathcal{S}_{i_h} are both sum-free. Next we recall that by construction any two numbers in \mathcal{U} differ by more than β , which is the sum of all numbers in \mathcal{S}_{i_h} . Finally the remark following (7) completes the argument.

We next consider Case 2. We define v by

$$(9) \quad \begin{aligned} v &= h_1/4 && \text{if 4 divides } h_1, \\ v &= [h_1/4] + 1 && \text{otherwise} \end{aligned}$$

and consider the v sets \mathcal{J}_{1+4j} ($j=0, 1, \dots, v-1$). We shall show that $\bigcup_{j=0}^{v-1} \mathcal{J}_{1+4j}$ is sum-free. To this end it suffices to show

(i) For each i ($i=0, 1, \dots, v-2$) the sum of all the numbers in $\bigcup_{j=i+1}^{v-1} \mathcal{J}_{1+4j}$ is less than any number in \mathcal{J}_{1+4i} .

(ii) The difference between any two distinct numbers belonging to the same \mathcal{J}_{1+4i} is less than any number in $\bigcup_{j=0}^{v-1} \mathcal{J}_{1+4j}$.

To see that (i) holds we note that the sum of all the numbers in $\bigcup_{j=i+1}^{v-1} \mathcal{J}_{1+4j}$ is $< 7M_i(1 + 2(1/2^4 + (1/2^4)^2 + \dots)) < 8M_i$ where M_i denotes largest number in $\mathcal{J}_{1+4(i+1)}$, whereas the smallest number in \mathcal{J}_{1+4i} is at least $8M_i$.

For the purpose of establishing (ii) we recall that, by construction, two distinct numbers in the same \mathcal{J}_{1+4j} differ by at most β whereas any number in $\bigcup_{j=0}^{v-1} \mathcal{J}_{1+4j}$ is larger than β , by the remark following (7).

Finally we see that Case 1 gives a sum-free set of cardinality at least

$$|\mathcal{U}| + |\mathcal{S}_{i_h}| > \frac{7}{6} \left(\frac{31}{35}\right) n^{1/2} = \frac{31}{30} n^{1/2},$$

and that Case 2 gives a sum-free set of cardinality at least

$$\left(\frac{h_1}{4}\right)7 > \frac{7}{4} \cdot \frac{3}{5} n^{1/2} = \frac{21}{20} n^{1/2},$$

on using (9) and (8).

In either case (1) is satisfied with $c=36/35$.

REFERENCE

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER 8, BRITISH COLUMBIA, CANADA