

A LOWER BOUND ON THE PERMANENT OF A (0, 1)-MATRIX

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ABSTRACT. The paper gives a lower bound for the permanent of a fully indecomposable (0, 1)-matrix with at least k ones in each row. The result extends those of H. Minc and P. Gibson.

Introduction. The computation of lower bounds for the permanent of a (0, 1)-matrix moved a step forward with the solution of Marshall Hall's conjecture by Richard Sinkhorn [5]. This was done by using, as a tool, the nearly decomposable matrix. Sinkhorn's argument rested heavily on the structure of this matrix which readily lends itself to inductive proofs. In [2] the structure of this matrix was simplified, thereby simplifying proofs for the above results.

In [1] and [3] more efficient counting methods were used to improve lower bounds. The result of this paper further develops counting methods. These then are used to extend the results of Minc and Gibson.

It is assumed that the reader is familiar with some of the language and notations of ([1], [2], [3], [4] and [5]).

Results. All matrices discussed in this work are $n \times n$ (0, 1)-matrices with $n \geq 3$. If $A = (a_{ij})$ is a matrix then $\sigma(A) = \sum_{i,j} a_{ij}$, and $S = \sigma(A) - (3n - 3)$.

The following two lemmas are vital to our arguments.

LEMMA 1. *If A is nearly decomposable then A contains at most $3n - 3$ ones [4, p. 184].*

LEMMA 2. *If A has at least $k \geq 3$ ones in each row then $S \geq (k - 3)n + 3$.*

PROOF. As $\sigma(A) \geq kn$ it follows that $\sigma(A) - (3n - 3) \geq kn - (3n - 3) = (k - 3)n + 3$.

Set $R = S - (k - 3)n$. For $k \geq 3$ Lemma 2 implies $R \geq 3$. Applying Lemma 1 it is seen that if A is fully indecomposable there are S ones in A which may be replaced by 0's yielding \bar{A} , with \bar{A} still fully indecomposable. Applying Lemma 2 it is seen that $(k - 3)n$ of these ones may be replaced in \bar{A} say n at a time yielding A_1, A_2, \dots, A_{k-3} so that A_i ($i = 1, 2, \dots, k - 3$)

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has at least $i+2$ ones in each row. This being done the remaining ones, R in number, may be replaced in A_{k-3} to reconstruct A .

The following notations are required for the work.

If $0 \leq R-1 < n$ set $R_1=R$, $R_2=1$. If $n \leq R-1$ set $R_1=n$, $R_2=R-n+1$.

We are now ready to give our bound. This bound improves the bound of H. Minc as well as that of P. Gibson.

THEOREM. *If A is fully indecomposable with $k \geq 3$ ones in each row then*

$$\begin{aligned} \text{per } A &\geq [\sigma(A) - 2n + 2] \\ &\quad + \sum_{i=2}^{k-3} (i! - 1)n + [(k-2)! - 1]R_1 + [(k-1)! - 1]R_2. \end{aligned}$$

PROOF. By replacing 1's in A with 0's a fully indecomposable matrix $A_0=(a_{ij}^{(0)})$ can be obtained so that $\sigma(A_0)=3n-3$. A_0 has at least two 1's in each row. Further, by [3] $\text{per } A_0 \geq \sigma(A_0) - 2n + 2$.

Pick n of the removed 1's from A (if $k \geq 4$) so that when 0's in A_0 are replaced by these 1's yielding $A_1=(a_{ij}^{(1)})$, A_1 has at least three ones in each row. Now

$$\text{per } A_1 \geq \text{per } A_0 + (n-1) + 2 \geq [\sigma(A_1) - 2n + 2] + 1.$$

The extra 1 in the second expression is obtained by noting that $n-1$ of the replaced 1's each lie on a positive diagonal and the final replaced 1 lies on two positive diagonals by Hall's inequality.

Pick n of the remaining removed 1's from A (if $k \geq 5$) so that when 0's in A_1 are replaced by these 1's yielding $A_2=(a_{ij}^{(2)})$, A_2 has at least four 1's in each row. Again by Hall's inequality

$$\begin{aligned} \text{per } A_2 &\geq \text{per } A_1 + 2!(n-1) + 3! \\ &\geq [\sigma(A_2) - 2n + 2] + 1 + (2! - 1)(n-1) + (3! - 1) \\ &\geq [\sigma(A_2) - 2n + 2] + (2! - 1)n + (3! - 1). \end{aligned}$$

By Lemma 2 then we see that for $r \leq k-3$ we have inductively $\text{per } A_r \geq [\sigma(A_r) - 2n + 2] + (2! - 1)n + \dots + (r! - 1)n + [(r+1)! - 1]$.

In particular for $r=k-3$,

$$\text{per } A_{k-3} \geq [\sigma(A_{k-3}) - 2n + 2] + \sum_{i=2}^{k-3} (i! - 1)n + [(k-2)! - 1].$$

We now argue two cases.

Case I. $R-1 < n$. Replace R_1-1 remaining ones into A_{k-3} to yield A_{k-2} thereby obtaining

$$\text{per } A_{k-2} \geq [\sigma(A) - 2n + 2] + \sum_{i=2}^{k-3} (i! - 1)n + [(k-2)! - 1]R_1.$$

As $R_1=R$ we replace the remaining 1 into A_{k-2} to yield A . Since A has at least k ones in each row, this 1 lies on $(k-1)!$ positive diagonals. Hence

$$\text{per } A \geq [\sigma(A) - 2n + 2] + \sum_{i=2}^{k-3} (i! - 1)n + [(k-2)! - 1]R_1 + [(k-1)! - 1]R_2.$$

Case II. $n \leq R-1$. Replace n remaining ones into A_{k-3} to yield A_{k-2} so that A_{k-2} has at least k ones in each row. Therefore we have

$$\text{per } A_{k-2} \geq [\sigma(A_{k-2}) - 2n + 2] + \sum_{i=2}^{k-3} (i! - 1)n + [(k-2)! - 1]n + [(k-1)! - 1].$$

Now replace all of the remaining ones into A_{k-2} to yield A thereby obtaining

$$\text{per } A \geq [\sigma(A) - 2n + 2] + \sum_{i=2}^{k-3} (i! - 1)n + [(k-2)! - 1]R_1 + [(k-1)! - 1]R_2.$$

This then yields the result of the theorem.

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