REGULAR ELEMENTS IN ALGEBRAIC GROUPS OF PRIME CHARACTERISTIC

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ABSTRACT. A result of Steinberg's on the existence of rational regular unipotent elements in quasi-split simple algebraic groups over fields of arbitrary characteristic is partially extended to the case of non-quasi-split groups.

1. Introduction. Let $k$ be a field of characteristic $p$ and let $G$ be a simple algebraic group defined over $k$. Let $S$ be a maximal $k$-split torus of $G$ and put $d = \dim Z_G(S)$. An element of $G_k$ is called $k$-split if it belongs to the $k$-split radical $S \cdot kU$ of a minimal $k$-parabolic subgroup $P = Z_G(S) \cdot kU$ of $G$. Such an element $g$ is called $k$-regular if $\dim Z_G(g) = d$. The purpose of this paper is to prove the theorem below which gives conditions under which $k$-regular unipotent elements exist. The case when $G$ is quasi-split over $k$ has been dealt with by Steinberg [10].

Theorem. Assume $G$ is not quasi-split, but $d < \dim G$. If the root system $\Sigma$ of $S$ in $G$ is reduced, $k$-regular unipotent elements exist provided $p$ is not one of the following types for the given $\Sigma$: $A_r, p|(r+1); B_r, p=2; C_r, p|2r; F_4, G_2, p=2, 3$. If $\Sigma$ is of type $BC_r$, $k$-regular unipotent elements exist provided $p \nmid 2(2r+1)$ and for $\alpha, 2\alpha \in \Sigma$ there exists a $k$-split 3-dimensional simple subgroup $K$ of $G$ normalized by $S$ with $\pm \alpha|(K \cap S)$ the roots of $K \cap S$ in $K$.

Remarks. (1) If $p$ is not "bad" for $G$ in the sense of Springer [8], every unipotent element of $G_k$ is $k$-split. See [9, p. 185].
(2) We do not know if our conditions on $p$ can be dropped. Various examples indicate they perhaps can be.
(3) For an indication of when $K$ exists see [7, pp. 121–125].

The proof is given in §3 below. §2 contains some elementary preliminaries on representations. Unexplained notation is that of [2].

2. Preliminaries. Let $\Sigma$ be a simple (reduced) root system in Euclidean space $E$, $\Delta$ a fundamental system defined by a linear order on $E$. Let
\( \mathcal{E} \subset E \) be the set of dominant weights defined by \( \Delta \). In addition to the linear order, we partially order \( E \) by \( \omega > \omega' \iff \omega - \omega' \) is a sum of positive roots. Let \( \Sigma_S \) denote the set of short roots in \( \Sigma \) (with \( \Sigma_S = \emptyset \) if there is only one root length). Put \( \Delta_S = \Delta \cap \Sigma_S, s = |\Delta_S|, r = |\Delta| \).

Let \( H \) be the simply simply connected \( k \)-split algebraic group with root system \( \Sigma \) relative to a fixed maximal \( k \)-split torus \( S \). Let \( \omega \in \mathcal{E} \) and let \( \rho : H \rightarrow GL(V) \) be the irreducible \( k \)-rational representation of dominant weight \( \omega \). We are interested in the \( \omega \) which have the properties that (1) the \( \omega < \omega \) in \( \mathcal{E} \) form an ordered sequence \( \omega = \omega_1 > \omega_2 > \cdots > \omega_n \), and (2) there is a unique sequence \( \{\alpha_1, \cdots, \alpha_{m_2}, \cdots, \alpha_{m_s} = \alpha_m\} \) of roots in \( \Delta \) such that for \( t \leq m \), \( \omega - \sum_{i=1}^{t} \alpha_i \) is conjugate under the Weyl group \( W \) of \( \Sigma \) to one of the \( \omega_t \) and equals \( \omega_t \) for \( t = m \). The following result summarizes some facts we will use.

**Proposition 1.** Suppose \( \omega \) satisfies (1) and (2) above. Assume also each \( \omega_t \) is a weight of \( S \) in \( V' \). Let \( \rho' : H \rightarrow GL(V') \) be a rational representation of \( H \). Let \( 0 \neq v \in V' \) be of weight \( \omega \) and fixed by \( U \), the unipotent radical \( R_u(B) \) of the Borel subgroup \( B \) defined by \( S \) and \( \Delta \). Let \( U \) be the submodule of \( V' \) generated by \( v \). Then if \( W' \) is a proper \( H \)-stable subspace of \( V' \), \( H \) acts trivially on \( W' \). Secondly, suppose that if \( 0 \neq v' \in V' \) has weight \( \omega' \) and is fixed by \( U \), then \( \omega = \omega_i \), some \( i \). Let \( \mathcal{E} = \{z \in V' : \rho(H)z = z\} \), \( V'' = V'/\mathcal{E} \), and let \( \rho'' : H \rightarrow GL(V'') \) be the induced representation. Then \( \rho'' \) is completely reducible.

**Proof.** Let \( B^- \) be the Borel subgroup defined by \( S \) and \( -\Delta \), and put \( U^- = R_u(B^-) \). Then \( S \cdot U^- \cdot U \) is dense in \( H \), so \( U = k^\rho'(H)v = k^\rho'(S \cdot U^-)v \). Hence \( \omega_t = \omega \) is the dominant weight of \( S \) in \( U \), and taking \( W' \) to be maximal proper \( H \)-stable, \( U'/W' \) is equivalent to \( V' \). Since the \( \omega_t \) are weights of \( S \) in \( V' \), they are weights of \( S \) in \( U \). If \( H \) does not act trivially on \( W' \), it follows from condition (1) on \( \omega \) that the dominant weight of \( S \) in \( W' \) equals some \( \omega_t \) and so the corresponding weight space \( U_{\omega_t} \) of \( U \) has dimension \( >1 \). But let \( u = u_{-m_2} \cdots u_{-m_s} \) where \( 1 \neq u_{-m_j} \in U_{-m_j} \). Then if \( \pi \) is the projection of \( U \) onto \( U_{\omega_t} \), \( \pi(u)v \) must span \( U_{\omega_t} \) by condition (2) on \( \omega \) and Lemme 1, Exposé 21 [3]. Hence, \( H \) acts trivially on \( W' \). For the second assertion of the proposition, we first remark that Lemme 1, Exposé 21 [3] shows that 0 is the only element of \( V'' \) fixed by \( H \). Let \( \{v_{ij}\}_{j \in J} \) be a basis for the \( \omega_t \)-weight space \( V''_{\omega_t} \) of \( V'' \). Then \( V''_{\omega_t} = \sum_{j \in J} V''_{\omega_t} \) is irreducible of dominant weight \( \omega_t \) by above, and the sum \( V''_{\omega_t} = \sum_{j \in J} V''_{\omega_t} \) is direct. Let \( \omega_t \) be the next member of the sequence \( \omega_1 > \omega_2 > \cdots > \omega_n \) for which there exists \( 0 \neq v'' \in V''_{\omega_t} \) with \( \rho''(U)v'' = v'' \). Let \( \{v_{2j}\}_{j \in J} \) be maximal linearly independent in \( V''_{\omega_t} \) with \( \rho''(U)v_{2j} = v_{2j} \) all \( j \). Then \( V''_{\omega_t} = k^\rho''(H)v_{2j} \) is irreducible of dominant weight \( \omega_t \) and the sum \( V''_{\omega_t} = \sum_{j \in J} V''_{\omega_t} \) is direct. Continuing in this way we obtain in \( q \)
steps, say, a completely reducible submodule \( \mathcal{V}'' \) of \( \mathcal{V}'' \) with the property that if \( v'' \in \mathcal{V}'' \) has weight \( \omega' \in \mathcal{D} \) and \( \rho''(U)v'' = v'' \) then \( v'' \in \mathcal{V}'' \). Hence \( H \) acts trivially on \( \mathcal{V}''/\mathcal{V}'' \), whence \( \mathcal{V}'' = \mathcal{V}'' \), proving the proposition. We also note that in the first part of the proof we have shown that if \( \lambda \) is a nonzero weight in \( \mathcal{V} \), then \( \dim \mathcal{V}_\lambda = 1 \) since \( \lambda \) is conjugate to some \( \omega_i \).

We assume familiarity with the construction of the irreducible representations of \( H \) from those of the corresponding complex simple Lie algebra by “reduction mod \( p' \)” [11, §12]. For the remainder of this section we determine conditions on \( p \) for which certain modules and maps remain irreducible and nonsingular in passage from characteristic 0 to characteristic \( p \). Let \( \mathcal{L} \) be the complex simple Lie algebra with root system \( \Sigma \), and let \( \mathcal{U} \) be the universal enveloping algebra for \( \mathcal{L} \). Let \( \{X_\alpha, H_\beta : \alpha \in \Sigma, \beta \in \Delta\} \) be a Chevalley basis for \( \mathcal{L} \) [11, p. 6], and let \( \mathcal{U}_Z, \mathcal{U}_Z^+, \mathcal{U}_Z^- \) be the \( Z \)-subalgebras of \( \mathcal{U} \) generated by \( X_\alpha^m/m! \) \( (m \in Z^+) \) for \( \alpha \in \Sigma, \Sigma^+, \Sigma^- \), respectively. Let \( \mu \) be the maximal root in \( \Sigma \), and if \( \Sigma_S \neq \varnothing \) let \( v \) be the maximal short root. For convenience we agree in the case of \( A_1 \) that \( \mu \) is both long and short (so \( \mu = v \)). In the following we take \( \omega \in \mathcal{D} \) to be one of three possibilities: \( \omega = \mu, v \), or in the case when \( \Sigma \) is of type \( B_k \) we allow \( \omega = 2v \). Conditions (1) and (2) preceding Proposition 1 are easily verified for these \( \omega \). Let \( \mathcal{V}^C \) be the irreducible \( \mathcal{L} \)-module of dominant weight \( \omega \). For \( 0 \neq v \in \mathcal{V}^C, M = \mathcal{U}_Z v \) is a \( \mathcal{U}_Z \)-stable lattice in \( \mathcal{V}^C \) [11, p. 17]. For a nonzero weight \( \lambda \) of \( \mathcal{L} \) in \( \mathcal{V}^C \), let \( v_\lambda \in M \) be so that \( Zv_\lambda = M \cap \mathcal{V}^C \). Let \( \Sigma' = \{\alpha \in \Sigma : \alpha \text{ is a weight in } \mathcal{V}^C\} = \Sigma \) or \( \Sigma_S \). Let \( M_0 \subset M \cap \mathcal{V}^C \) be the \( Z \)-span of the \( X_{-\gamma}v_\gamma \) for \( \gamma \in \Delta' = \Delta \cap \Sigma' \). The nonzero weights in \( \mathcal{V}^C \) are roots and, if \( \omega = 2v \), twice short roots. Thus, for \( \alpha \neq \beta \in \Delta' \), \( X_\beta X_\alpha v_\alpha = -\alpha(H_\beta)v_\alpha \). Hence, once for a given \( \gamma \in \Delta' \), \( v_\gamma \) is fixed (it is unique up to changes in sign), since \( M \) is \( \mathcal{U}_Z \)-stable, the other \( \nu_\beta, \beta \in \Delta' \), are uniquely determined such that for \( \alpha, \beta \in \Delta' \), \( X_\beta X_\alpha v_\alpha = \langle \alpha, \beta \rangle v_\beta \) where \( \langle \alpha, \beta \rangle = \alpha(H_\beta) \) (with \( \langle \beta, \alpha \rangle \) then equal to \( \beta(H_\alpha) \) or \( \alpha(H_\beta) \), respectively), except if \( \alpha = 2v \) and \( \alpha = \beta \in \Sigma_S \) when \( \langle \alpha, \alpha \rangle = 6 \) because of the representation theory of the simple Lie algebra \( CX_\alpha + CH_\alpha + CX_\alpha \) acting irreducibly on the 5-dimensional space it generates from \( v_{2a} \) [4, p. 83]. Given \( v = \sum_{\alpha \in \Delta'} c_\alpha X_\alpha v_\alpha \in k \otimes M_0 \), in order that \( X_\beta v = 0 \) for all \( \beta \in \Delta' \) we must therefore have \( \sum_{\alpha \in \Delta'} c_\alpha(\alpha, \beta) = 0, \beta \in \Delta' \). The determinant of the matrix \( C' \) of this system of equations is by above the determinant of the Cartan matrix of \( \Delta' \), except when \( \omega = 2v \) when it is easily calculated to be \( 2(2r+1) \).

Consider the irreducible \( H \)-module \( \mathcal{V} \) of dominant weight \( \omega \) obtained from the \( \mathcal{L} \)-module \( \mathcal{V}^C \). Given \( \omega_i \) \( (i > 1) \), \( \lambda = \omega_i + \alpha_m \) is \( W \)-conjugate to \( \omega_j \) by condition (2) on \( \omega \), and \( \omega_j > \omega_i \) since \( \lambda > \omega_i \). Then

\[
X_{\alpha_m}v_{\omega_i} = X_{\alpha_m}X_{-\alpha_m}v_{\lambda} = H_{\alpha_m}v_{\lambda} = \lambda(H_{\alpha_m})v_{\lambda}
\]

since \( \lambda + \alpha_m \) is not a weight in \( \mathcal{V}^C \). Thus, if \( p \neq \lambda(H_{\alpha_m}) \) \( (= -2 \) if \( i = 2 \) with
\[ \omega = \mu \text{ in } B_r, C_r, F_4, \text{ or if } i=2, 3 \text{ with } \omega = 2\nu \text{ in } B_r; \omega = -3 \text{ if } i=2, \omega = \mu \text{ in } G_2 \] 
\[ \omega_i \text{ is a weight of } S \text{ in } \mathcal{Y} \text{ by induction on } i. \] 
If in addition \( p \nmid \det C' \), we have also shown \( \dim \mathcal{Y}_0 = |\Delta'| = r \) or \( s \).

For \( \alpha \in \Delta \), put \( n_{\alpha} = \text{Exp}(X_{\alpha})\text{Exp}(-X_{-\alpha})\text{Exp}(X_{\alpha}) \), an automorphism of \( \mathcal{Y}_G \) with \( n_{\alpha}M = M \). We order \( \Delta \) so that the first \( q \) roots \( \alpha_1, \cdots, \alpha_q \) are orthogonal as are the last \( r-q \) roots. The Coxeter transformation \( w_{\Delta}, \) acting on \( \mathcal{Y}_0' \) is given by \( n_{w} = n_{\alpha_q} \cdots n_{\alpha_1} \). For \( \gamma \in \Delta' \), \( \alpha \in \Delta \), 
\[ n_{\alpha}X_{-\gamma}v_{\gamma} = X_{-\gamma}v_{\gamma} \text{ if } \gamma \notin \Delta' \text{ (since otherwise from the form of } n_{\alpha} \text{ we would have } \gamma \in \Sigma' \), and 
\[ n_{\alpha}X_{-\gamma}v_{\gamma} = X_{-\gamma}v_{\gamma} - \langle \gamma, \alpha \rangle' X_{-\alpha}v_{\alpha} \text{ for } \gamma \in \Delta' \text{ where } \langle \gamma, \alpha \rangle' = \langle \gamma, \alpha \rangle \text{ if } \alpha \neq \gamma \text{ and } \langle \alpha, \alpha \rangle' = 2. \]

For \( i \leq q \)
\[ (1 - n_{w})X_{-\alpha_i}v_{\alpha_i} = 2X_{-\alpha_i}v_{\alpha_i} - \sum_{j > q, \alpha_j \in \Delta'} \langle \alpha_i, \alpha_j \rangle X_{-\alpha_j}v_{\alpha_j}, \]
while for \( i > q \),
\[ (1 - n_{w})X_{-\alpha_i}v_{\alpha_i} = 2X_{-\alpha_i}v_{\alpha_i} - \sum_{j \leq q, \alpha_j \in \Delta'} \langle \alpha_i, \alpha_j \rangle n_{w}X_{-\alpha_j}v_{\alpha_j}, \]
\[ = 2X_{-\alpha_i}v_{\alpha_i} + \sum_{j \leq q} \langle \alpha_i, \alpha_j \rangle \left( X_{-\alpha_j}v_{\alpha_j} - \sum_{k > q, \alpha_k \in \Delta'} \langle \alpha_j, \alpha_k \rangle X_{-\alpha_k}v_{\alpha_k} \right). \]

Substituting the conditions \((*)\) into \((**)\) we transform the matrix obtained for \(1 - n_{w}\) into \((|c'_{ij}|)\) where \((c'_{ij})\) is the Cartan matrix of \( \Delta' \). A simple induction argument shows \( \det(|c'_{ij}|) = \det(c_{ij}) \).

Collecting together the various restrictions on \( p \), we get

**Proposition 2.** Let \( \omega = \mu, \nu \), or in the case of \( B_r \) possibly \( 2\nu \). For the following \( \Sigma \) suppose \( p \) is as indicated: \( A_r, p \nmid (r+1) \); \( C_r, p \nmid 2r \); \( B_r, p \nmid 2 \) if \( \omega < 2\nu \) and \( p \nmid 2(2r+1) \) if \( \omega = 2\nu \); \( F_4, G_2, p \nmid 2, 3 \). Let \( p: H \rightarrow GL(\mathcal{Y}) \) be the irreducible representation of \( H \) of dominant weight \( \omega \). Then if \( \omega' < \omega \), \( \omega' \in \Delta \), \( \omega' \) is a weight of \( S \) in \( \mathcal{Y} \). Also, \( \dim \mathcal{Y}_0' = r \) except if \( \omega = \nu \) when it equals \( s \). Finally, \( 1 \) is not an eigenvalue of \( \rho(n_{w}) \mathcal{Y}_0' \) if \( n_{w} \in N_H(S) \) represents the Coxeter transformation \( w \).

We also need

**Proposition 3.** The Coxeter transformation \( w \) defined above decomposes \( \Sigma \) into \( r \) orbits of roots and \( \Sigma_{G} \) into \( s \) orbits of roots.

**Proof.** As is well known [5, Corollary 8.2], \( w \) decomposes \( \Sigma \) into \( r \) orbits each with \( h = 1 + \text{height } \mu \) elements. Therefore, the second assertion follows since \( |\Sigma_{G}| = (2r, B_r; 2r(r-1), C_r; 26, F_4; 6, G_2) \) divided by \( h = (2r, B_r, C_r; 13, F_4; 6, G_2) \) equals \( s = (1, B_r, G_2; r-1, C_r; 2, F_4) \).

**3. Proof of the Theorem.** Put \( G^0_S = Z_G(S), G = \text{Lie}(G), G^0 = \text{Lie}(G^0_S) = Z_{G_S}(S). \) Let \( \Delta \) be a fundamental system for \( \Delta \), and for \( \alpha \in \Delta \), let \( 1 \neq u_\alpha \in U(a, k). \) Define \( u = \prod_{\alpha \in \Delta} u_\alpha \) in some order. We assume first that the root
system \(\Sigma\) of \(S\) is reduced. Borel and Tits [2, §7] have constructed a \(k\)-split simple subgroup \(H \supseteq S\) such that the \(u_\alpha \in H\), and so that the root system \(\Sigma_H\) of \(S\) in \(H\) identifies with \(\Sigma\). Let \(\rho : H \to GL(\mathfrak{g})\) be the adjoint representation of \(H\) in \(\mathfrak{g}\). Let \(\mathcal{L} = Z_{\mathfrak{g}}(H) \subseteq \mathfrak{g}'_0\), and put \(\mathcal{G}' = \mathfrak{g}'/\mathcal{L}\) with \(\rho' : H \to GL(\mathfrak{g}')\) the induced representation. We restrict \(\rho\) to not being one of the types listed in the theorem, and Propositions 1 and 2 show \(\rho'\) is completely reducible. Let \(d_\mu\) and \(d_\nu\) be the dimensions respectively of \(U(\mu)\) and \(U(\nu)\). Then \(\mathfrak{g}\) decomposes as \(\mathcal{V}_1 \oplus \mathcal{V}_2\) where \(\mathcal{V}_1\) (resp. \(\mathcal{V}_2\)) is the direct sum of \(d_\mu\) (resp. \(d_\nu - d_\mu\)) irreducible \(H\)-submodules of dominant weight \(\mu\) (resp. \(\nu\)). It is clear that \(\dim Z_{\mathfrak{g}}(u) \leq \dim Z_{\mathfrak{g}}(u) + \dim \mathcal{L}\). Since \(\dim Z_{\mathfrak{g}}(u) \geq \dim Z_{\mathfrak{g}}(u) \biggeq d\) trivially, it suffices to show \(\dim Z_{\mathfrak{g}}(u) + \dim \mathcal{L} \leq d\). We have \(Z_{\mathfrak{g}}(u) = Z_{\mathfrak{g}}(u) \oplus Z_{\mathfrak{g}}(u)\). Let \(\mathcal{V}_1\) and \(\mathcal{V}_2\) be irreducible \(H\)-modules of dominant weights \(\mu\) and \(\nu\) respectively. By Proposition 2 if \(n_u \in H\) represents the Coxeter transformation defined earlier, \(1 - n_u\) is nonsingular on the zero weight spaces \((\mathcal{W}_1)_0\). By [10, Lemma 4.5], \(u\) is conjugate in \(H\) to an element \(y\) of the form \(bn_u, b \in B\). By Proposition 3, \(w\) decomposes \(\Sigma\) into \(r = |\Delta|\) orbits of roots and \(\Sigma_{\mathfrak{g}}\) into \(s = |\Delta_{\mathfrak{g}}|\) orbits of roots, so that exactly as in [10, Lemma 4.3], \(\dim Z_{\mathfrak{g}}(y) \leq r\) and \(\dim Z_{\mathfrak{g}}(y) \leq s\). Hence, \(\dim Z_{\mathfrak{g}}(y) \leq d_\mu r\) and \(\dim Z_{\mathfrak{g}}(y) \leq (d_\nu - d_\mu)s\). By Proposition 2, \(r = \dim (\mathcal{W}_1)_0\) and \(s = \dim (\mathcal{W}_2)_0\), so \(\dim Z_{\mathfrak{g}}(u) \leq d_\mu r + (d_\nu - d_\mu)s = \dim Z_{\mathfrak{g}}(S)\). Since \(\mathcal{L} \subseteq \mathfrak{g}'_0\) we get finally \(\dim Z_{\mathfrak{g}}(u) + \dim \mathcal{L} \leq \dim \mathfrak{g}'_0 = d\), as desired.

Next, assume \(\Sigma\) is of type \(B_r, P_{2r+1}\), and the subgroups \(K\) exist. Then [6, Proposition 5.1], there exists a simple \(k\)-split subgroup \(H \supseteq S\) of type \(B_r\). Put \(d_\mu = \dim U(\mu), d_\nu = \dim U(\nu)\). Define \(u\) as before with \(u \in H\). Let \(\rho_i : H \to GL(\mathcal{W}_i), i = 1, 2, 3\), be irreducible of dominant weight \(2\nu, \mu, \nu\), respectively. \(H\) acts completely reducibly on \(\mathfrak{g}'\) as defined above, and \(\mathcal{G}' = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3\) where \(\mathcal{V}_i\) is the direct sum of \(c_i\) copies of \(\mathcal{W}_i (c_1 = d_\nu, c_2 = d_\mu - d_\nu, c_3 = d_\nu - d_\mu)\). As before, \(\dim Z_{\mathfrak{g}}(y) \leq r\) and \(\dim Z_{\mathfrak{g}}(y) \leq s\), so to finish we must show \(\dim Z_{\mathfrak{g}}(u) \leq r\). Put \(\Delta = \{\alpha_1, \ldots, \alpha_r\}\) with \(\alpha_i\) connected to \(\alpha_{i+1}\) (\(i < r\), \(\alpha_i\) long, \(\alpha_r\) short. If \(\lambda = \sum n_i \alpha_i \in Z_{\mathfrak{g}}\Delta\), call \(h(\lambda) = \sum n_i\) the height of \(\lambda\). Let \(\mathcal{W}_i\) be the subspace of \(\mathcal{W}_i\) spanned by vectors of weight of height \(i\), and let \(\pi_i\) be the projection of \(\mathcal{W}_i\) onto \((\mathcal{W}_i)_i\). We say a weight \(\alpha\) is related to a weight \(\lambda\) if \(\lambda = \lambda' + \alpha\) for some \(\alpha \in \Delta\). If \(\lambda'\) is so related to \(\lambda \neq 0\), since \(\rho' \neq 2\), a slight extension of the argument on p. 6 (or Proposition 2.2, [6]) shows \(\pi_j \rho_j(u) (\mathcal{W}_i)_i = (\mathcal{W}_j)_i\), \(\pi_j \) the projection of \(\mathcal{W}_i\) onto \((\mathcal{W}_j)_i\). Define \(\beta_i = \alpha_i\) if \(1 \leq i \leq r\) and \(= \alpha_{2r-i+1}\) if \(r < i \leq 2r\). For \(1 \leq j \leq 2r\), put \(\gamma_{j,t} = \beta_i + \cdots + \beta_{2r-t+1}\) for \(1 \leq t \leq n_j = \min\{r, r-j+2\}\), the weights of heights \(j\). If \(j > 1\), \(\gamma_{j-1,1}\) is related only to \(\gamma_{j,1}\) and if \(r > 1\), \(\gamma_{j-t,1}\) is related only to \(\gamma_{j,t,1}\) and \(\gamma_{j,t}\) (if \(t \leq n_j\)). Hence, if \(j > 1\), \(\pi_j \rho_j(u)\) maps \((\mathcal{W}_i)_j\) onto \((\mathcal{W}_j)_j\). Similarly, if \(r > 1\), \(\pi_j \rho_j(u)\) maps \((\mathcal{W}_i)_j\) onto \((\mathcal{W}_j)_r\). The proof
that $\dim(W_1)_0 = r$ (Proposition 2) shows $\pi_j \rho_j(u)$ maps $(W_1)_{j-1}$ injectively into $(W_1)_j, j=0,1$. Hence, $\dim Z_{W_1}(u) \leq \sum_{j \geq 1} \dim(W_1)_j - \dim(W_1)_{j+1} = \dim(W_1)_1 = r$, as desired.

Finally, we should note the characteristic zero case is much simpler, a $k$-regular unipotent element being obtained by taking the exponential of a $k$-principal nilpotent element in $G$. See [1] and [5].

**Bibliography**


11. ———, *Lectures on Chevalley groups*, Yale University, New Haven, Conn., 1968 (mimeographed).

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