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REGULAR ELEMENTS IN ALGEBRAIC GROUPS OF PRIME CHARACTERISTIC

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ABSTRACT. A result of Steinberg's on the existence of rational regular unipotent elements in quasi-split simple algebraic groups over fields of arbitrary characteristic is partially extended to the case of non-quasi-split groups.

1. **Introduction.** Let k be a field of characteristic p and let G be a simple algebraic group defined over k. Let S be a maximal k-split torus of G and put $d=\dim Z_G(S)$. An element of G_k is called *k-split* if it belongs to the k-split radical $S \cdot {}_kU$ of a minimal k-parabolic subgroup $P=Z_G(S) \cdot {}_kU$ of G. Such an element g is called *k-regular* if $\dim Z_G(g)=d$. The purpose of this paper is to prove the theorem below which gives conditions under which k-regular unipotent elements exist. The case when G is quasi-split over k has been dealt with by Steinberg [**10**].

THEOREM. *Assume G is not quasi-split, but $d<\dim G$. If the root system ${}_k\Sigma$ of S in G is reduced, k-regular unipotent elements exist provided p is not one of the following types for the given ${}_k\Sigma$: $A_r, p\mid(r+1)$; $B_r, p=2$; $C_r, p\mid 2r$; F_4, $G_2, p=2, 3$. If ${}_k\Sigma$ is of type BC_r, k-regular unipotent elements exist provided $p\nmid 2(2r+1)$ and for $\alpha, 2\alpha \in {}_k\Sigma$ there exists a k-split 3-dimensional simple subgroup K of G normalized by S with $\pm\alpha|(K\cap S)$ the roots of $K\cap S$ in K.*

REMARKS. (1) If p is not "bad" for G in the sense of Springer [**8**], every unipotent element of G_k is k-split. See [**9**, p. 185].

(2) We do not know if our conditions on p can be dropped. Various examples indicate they perhaps can be.

(3) For an indication of when K exists see [**7**, pp. 121–125].

The proof is given in §3 below. §2 contains some elementary preliminaries on representations. Unexplained notation is that of [**2**].

2. **Preliminaries.** Let Σ be a simple (reduced) root system in Euclidean space E, Δ a fundamental system defined by a linear order on E. Let

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$\mathscr{D} \subset E$ be the set of dominant weights defined by Δ. In addition to the linear order, we partially order E by $\omega > \omega' \Leftrightarrow \omega - \omega'$ is a sum of positive roots. Let Σ_S denote the set of short roots in Σ (with $\Sigma_S = \varnothing$ if there is only one root length). Put $\Delta_S = \Delta \cap \Sigma_S$, $s = |\Delta_S|$, $r = |\Delta|$.

Let H be the simple simply connected k-split algebraic group with root system Σ relative to a fixed maximal k-split torus S. Let $\omega \in \mathscr{D}$ and let $\rho: H \to GL(\mathscr{V})$ be the irreducible k-rational representation of dominant weight ω. We are interested in the ω which have the properties that (1) the $\omega' < \omega$ in $\mathscr{D}$ form an ordered sequence $\omega = \omega_1 > \omega_2 > \cdots > \omega_n$, and (2) there is a unique sequence $\{\alpha_1, \cdots, \alpha_{m_2}, \cdots, \alpha_{m_3}, \cdots, \alpha_{m_n} = \alpha_m\}$ of roots in Δ such that for $t \leqq m$, $\omega - \sum_{j=1}^{t} \alpha_i$ is conjugate under the Weyl group W of Σ to one of the ω_j and equals ω_h for $t = m_h$. The following result summarizes some facts we will use.

PROPOSITION 1. *Suppose ω satisfies (1) and (2) above. Assume also each ω_i is a weight of S in $\mathscr{V}$. Let $\rho': H \to GL(\mathscr{V}')$ be a rational representation of H. Let $0 \neq v \in \mathscr{V}'$ be of weight ω and fixed by U, the unipotent radical $R_u(B)$ of the Borel subgroup B defined by S and Δ. Let $\mathscr{U}$ be the submodule of $\mathscr{V}'$ generated by v. Then if $\mathscr{W}$ is a proper H-stable subspace of $\mathscr{U}$, H acts trivially on $\mathscr{W}$. Secondly, suppose that if $0 \neq v' \in \mathscr{V}'$ has weight $\omega' \in \mathscr{D}$ and is fixed by U, then $\omega' = \omega_i$, some i. Let $\mathscr{Z} = \{z \in \mathscr{V}' : \rho(H)z = z\}$, $\mathscr{V}'' = \mathscr{V}'/\mathscr{Z}$, and let $\rho'': H \to GL(\mathscr{V}'')$ be the induced representation. Then ρ'' is completely reducible.*

PROOF. Let B^- be the Borel subgroup defined by S and $-\Delta$, and put $U^- = R_u(B^-)$. Then $S \cdot U^- \cdot U$ is dense in H, so $\mathscr{U} = k\rho'(H)v = \bar{k}\rho'(S \cdot U^-)v$. Hence $\omega_1 = \omega$ is the dominant weight of S in $\mathscr{U}$, and taking $\mathscr{W}$ to be maximal proper H-stable, $\mathscr{U}/\mathscr{W}$ is equivalent to $\mathscr{V}$. Since the ω_i are weights of S in $\mathscr{V}$, they are weights of S in $\mathscr{U}$. If H does not act trivially on $\mathscr{W}$, it follows from condition (1) on ω that the dominant weight of S in $\mathscr{W}$ equals some ω_j and so the corresponding weight space $\mathscr{U}_{\omega_j}$ of $\mathscr{U}$ has dimension >1. But let $u = u_{-\alpha_{m_j}} \cdots u_{-\alpha_1}$ where $1 \neq u_{-\alpha_i} \in U_{-\alpha_i}$. Then if π is the projection of $\mathscr{U}$ onto $\mathscr{U}_{\omega_j}$, $\pi(\rho'(u)v)$ must span $\mathscr{U}_{\omega_j}$ by condition (2) on ω and Lemme 1, Exposé 21 [3]. Hence, H acts trivially on $\mathscr{W}$. For the second assertion of the proposition, we first remark that Lemme 1, Exposé 21 [3] shows that 0 is the only element of $\mathscr{V}''$ fixed by H. Let $\{v_{1j}\}_{j \in J_1}$ be a basis for the ω_1-weight space $\mathscr{V}''_{\omega_1}$ of $\mathscr{V}''$. Then $\mathscr{V}''_{1j}\bar{k}\rho''(H)v_{1j}$ is irreducible of dominant weight ω_1 by above, and the sum $\mathscr{V}''_1 = \sum_{j \in J_1} \mathscr{V}''_{1j}$ is direct. Let ω_t be the next member of the sequence $\omega_1 > \omega_2 > \cdots > \omega_n$ for which there exists $0 \neq v'' \in \mathscr{V}''_{\omega_t}$ with $\rho''(U)v'' = v''$. Let $\{v_{2j}\}_{j \in J_2}$ be maximal linearly independent in $\mathscr{V}''_{\omega_t}$ with $\rho''(U)v_{2j} = v_{2j}$ all j. Then $\mathscr{V}''_{2j} = \bar{k}\rho''(H)v_{2j}$ is irreducible of dominant weight ω_t and the sum $\mathscr{V}''_2 = \mathscr{V}''_1 + \sum_{j \in J_2} \mathscr{V}''_{2j}$ is direct. Continuing in this way we obtain in q

steps, say, a completely reducible submodule $\mathscr{V}''_q$ of $\mathscr{V}''$ with the property that if $v'' \in \mathscr{V}''$ has weight $\omega' \in \mathscr{D}$ and $\rho''(U)v''=v''$ then $v'' \in \mathscr{V}''_q$. Hence H acts trivially on $\mathscr{V}''/\mathscr{V}''_q$, whence $\mathscr{V}''=\mathscr{V}''_q$, proving the proposition. We also note that in the first part of the proof we have shown that if λ is a nonzero weight in $\mathscr{V}$, then $\dim \mathscr{V}_\lambda=1$ since λ is conjugate to some ω_i.

We assume familiarity with the construction of the irreducible representations of H from those of the corresponding complex simple Lie algebra by "reduction mod p" [**11**, §12]. For the remainder of this section we determine conditions on p for which certain modules and maps remain irreducible and nonsingular in passage from characteristic 0 to characteristic p. Let $\mathscr{L}$ be the complex simple Lie algebra with root system Σ, and let $\mathscr{U}$ be the universal enveloping algebra for $\mathscr{L}$. Let $\{X_\alpha, H_\beta:\alpha \in \Sigma, \beta \in \Delta\}$ be a Chevalley basis for $\mathscr{L}$ [**11**, p. 6], and let $\mathscr{U}_Z$, $\mathscr{U}_Z^+$, $\mathscr{U}_Z^-$ be the Z-subalgebras of $\mathscr{U}$ generated by $X_\alpha^m/m!$ $(m \in Z^+)$ for $\alpha \in \Sigma$, Σ^+, Σ^-, respectively. Let μ be the maximal root in Σ, and if $\Sigma_S \neq \varnothing$ let ν be the maximal short root. For convenience we agree in the case of A_1 that μ is both long and short (so $\mu=\nu$). In the following we take $\omega \in \mathscr{D}$ to be one of three possibilities: $\omega=\mu$, ν, or in the case when Σ is of type B_r we allow $\omega=2\nu$. Conditions (1) and (2) preceding Proposition 1 are easily verified for these ω. Let $\mathscr{V}^C$ be the irreducible $\mathscr{L}$-module of dominant weight ω. For $0\neq v \in \mathscr{V}^C_\omega$, $M=\mathscr{U}_Z^- v$ is a $\mathscr{U}_Z$-stable lattice in $\mathscr{V}^C$ [**11**, p. 17]. For a nonzero weight λ of $\mathscr{L}$ in $\mathscr{V}^C$, let $v_\lambda \in M$ be so that $Zv_\lambda=M \cap \mathscr{V}^C_\lambda$. Let $\Sigma'=\{\alpha \in \Sigma:\alpha \text{ is a weight in } \mathscr{V}^C\}=\Sigma$ or Σ_S. Let $M_0' \subset M \cap \mathscr{V}^C_0$ be the Z-span of the $X_{-\gamma}v_\gamma$ for $\gamma \in \Delta'=\Delta \cap \Sigma'$. The nonzero weights in $\mathscr{V}^C$ are roots and, if $\omega=2\nu$, twice short roots. Thus, for $\alpha\neq\beta$ in Δ', $X_{-\beta}X_\beta v_\alpha= -\alpha(H_\beta)v_\alpha$. Hence, once for a given $\gamma \in \Delta'$, v_γ is fixed (it is unique up to changes in sign), since M is $\mathscr{U}_Z$-stable, the other v_β, $\beta \in \Delta'$, are uniquely determined such that for $\alpha, \beta \in \Delta'$, $X_\beta X_{-\alpha}v_\alpha=\langle\alpha, \beta\rangle v_\beta$ where $\langle\alpha, \beta\rangle= \alpha(H_\beta)$ or $\beta(H_\alpha)$ (with $\langle\beta, \alpha\rangle$ then equal to $\beta(H_\alpha)$ or $\alpha(H_\beta)$, respectively), except if $\omega=2\nu$ and $\alpha=\beta \in \Sigma_S$ when $\langle\alpha, \alpha\rangle=6$ because of the representation theory of the simple Lie algebra $CX_\alpha+CH_\alpha+CX_{-\alpha}$ acting irreducibly on the 5-dimensional space it generates from $v_{2\alpha}$ [**4**, p. 83]. Given $v= \sum_{\alpha\in\Delta'} c_\alpha X_{-\alpha}v_\alpha \in \bar{k}\otimes M_0'$, in order that $X_\beta v=0$ for all $\beta \in \Delta'$ we must therefore have $\sum_{\alpha\in\Delta'} c_\alpha\langle\alpha, \beta\rangle=0$, $\beta \in \Delta'$. The determinant of the matrix C' of this system of equations is by above the determinant of the Cartan matrix of Δ', except if $\omega=2\nu$ when it is easily calculated to be $2(2r+1)$.

Consider the irreducible H-module $\mathscr{V}$ of dominant weight ω obtained from the $\mathscr{L}$-module $\mathscr{V}^C$. Given ω_i $(i>1)$, $\lambda=\omega_i+\alpha_{m_i}$ is W-conjugate to ω_j by condition (2) on ω, and $\omega_j>\omega_i$ since $\lambda>\omega_i$. Then

$$X_{\alpha_{m_i}}v_{\omega_i} = X_{\alpha_{m_i}}X_{-\alpha_{m_i}}v_\lambda = H_{\alpha_{m_i}}v_\lambda = \lambda(H_{\alpha_{m_i}})v_\lambda$$

since $\lambda+\alpha_{m_i}$ is not a weight in $\mathscr{V}^C$. Thus, if $p\nmid\lambda(H_{\alpha_{m_i}})$ $(=-2$ if $i=2$ with

$\omega=\mu$ in B_r, C_r, F_4, or if $i=2, 3$ with $\omega=2\nu$ in B_r; $=-3$ if $i=2$, $\omega=\mu$ in G_2) ω_i is a weight of S in $\mathscr{V}$ by induction on i. If in addition $p\nmid\det C'$, we have also shown $\dim \mathscr{V}_0=|\Delta'|=r$ or s.

For $\alpha\in\Delta$, put $n_\alpha=\operatorname{Exp}(X_\alpha)\operatorname{Exp}(-X_{-\alpha})\operatorname{Exp}(X_\alpha)$, an automorphism of $\mathscr{V}^C$ with $n_\alpha M=M$. We order Δ so that the first q roots $\alpha_1,\cdots,\alpha_q$ are orthogonal as are the last $r-q$ roots. The Coxeter transformation $w\Delta$, $w_{\alpha_r}\cdots w_{\alpha_1}$ acting on $\mathscr{V}_0^C$ is given by $n_w=n_{\alpha_r}\cdots n_{\alpha_1}$. For $\gamma\in\Delta'$, $\alpha\in\Delta$, $n_\alpha X_{-\gamma}v_\gamma=X_{-\gamma}v_\gamma$ if $\alpha\notin\Delta'$ (since otherwise from the form of n_α we would have $\alpha\in\Sigma'$), and $n_\alpha X_{-\gamma}v_\gamma=X_{-\gamma}v_\gamma-\langle\gamma,\alpha\rangle' X_{-\alpha}v_\alpha$ for $\alpha\in\Delta'$ where $\langle\gamma,\alpha'\rangle=\langle\gamma,\alpha\rangle$ if $\alpha\neq\gamma$ and $\langle\alpha,\alpha'\rangle=2$. For $i\leqq q$

$$(*)\qquad (1-n_w)X_{-\alpha_i}v_{\alpha_i}=2X_{-\alpha_i}v_{\alpha_i}-\sum_{j>q;\alpha_j\in\Delta'}\langle\alpha_i,\alpha_j\rangle' X_{-\alpha_j}v_{\alpha_j},$$

while for $i>q$,

$$\begin{aligned}(**)\qquad &(1-n_w)X_{-\alpha_i}v_{\alpha_i}\\ &=2X_{-\alpha_i}v_{\alpha_i}-\sum_{j\leqq q;\alpha_j\in\Delta'}\langle\alpha_i,\alpha_j\rangle' n_w X_{-\alpha_j}v_{\alpha_j}\\ &=2X_{-\alpha_i}v_{\alpha_i}+\sum_{j\leqq q}\langle\alpha_i,\alpha_j\rangle'\Big(X_{-\alpha_j}v_{\alpha_j}-\sum_{k>q;\alpha_k\in\Delta'}\langle\alpha_j,\alpha_k\rangle' X_{-\alpha_k}v_{\alpha_k}\Big).\end{aligned}$$

Substituting the conditions (*) into (**) we transform the matrix obtained for $1-n_w$ into $(|c'_{ij}|)$ where (c'_{ij}) is the Cartan matrix of Δ'. A simple induction argument shows $\det(|c'_{ij}|)=\det(c'_{ij})$.

Collecting together the various restrictions on p, we get

Proposition 2. *Let $\omega=\mu$, ν, or in the case of B_r possibly 2ν. For the following Σ suppose p is as indicated: A_r, $p\nmid(r+1)$; C_r, $p\nmid 2r$; B_r, $p\neq 2$ if $\omega<2\nu$ and $p\nmid 2(2r+1)$ if $\omega=2\nu$; F_4, G_2, $p\neq 2, 3$. Let $\rho: H\to GL(\mathscr{V})$ be the irreducible representation of H of dominant weight ω. Then if $\omega'<\omega$, $\omega'\in\mathscr{D}$, ω' is a weight of S in $\mathscr{V}$. Also,* $\dim\mathscr{V}_0=r$ *except if $\omega=\nu$ when it equals s. Finally, 1 is not an eigenvalue of $\rho(n_w)|\mathscr{V}_0$ if $n_w\in N_H(S)$ represents the Coxeter transformation w.*

We also need

Proposition 3. *The Coxeter transformation w defined above decomposes Σ into r orbits of roots and Σ_S into s orbits of roots.*

Proof. As is well known [**5**, Corollary 8.2], w decomposes Σ into r orbits each with $h=1+\text{height }\mu$ elements. Therefore, the second assertion follows since $|\Sigma_S|$ $(=2r, B_r; 2r(r-1), C_r; 26, F_4; 6, G_2)$ divided by h $(=2r, B_r, C_r; 13, F_4; 6, G_2)$ equals s $(=1, B_r, G_2; r-1, C_r; 2, F_4)$.

3. **Proof of the Theorem.** Put $G_0^r=Z_G(S)$, $\mathfrak{G}=\operatorname{Lie}(G)$, $\mathfrak{G}_0^r=\operatorname{Lie}(G_0^r)=Z_{\mathfrak{G}}(S)$. Let ${}_k\Delta$ be a fundamental system for ${}_k\Sigma$, and for $\alpha\in{}_k\Delta$, let $1\neq u_\alpha\in U_{(\alpha),k}$. Define $u=\prod_{\alpha\in_k\Delta}u_\alpha$ in some order. We assume first that the root

system ${}_k\Sigma$ of S is reduced. Borel and Tits [**2**, §7] have constructed a k-split simple subgroup $H\supset S$ such that the $u_\alpha \in H$, and so that the root system Σ_H of S in H identifies with ${}_k\Sigma$. Let $\rho: H\to GL(\mathfrak{G})$ be the adjoint representation of H in $\mathfrak{G}$. Let $\mathscr{Z}=Z_{\mathfrak{G}}(H)\subset\mathfrak{G}_0^r$, and put $\mathfrak{G}'=\mathfrak{G}/\mathscr{Z}$ with $\rho': H\to GL(\mathfrak{G}')$ the induced representation. We restrict p to not being one of the types listed in the theorem, and Propositions 1 and 2 show ρ' is completely reducible. Let d_μ and d_ν be the dimensions respectively of $U_{(\mu)}$ and $U_{(\nu)}$. Then $\mathfrak{G}$ decomposes as $\mathscr{V}_1\oplus\mathscr{V}_2$ where $\mathscr{V}_1$ (resp. $\mathscr{V}_2$) is the direct sum of d_μ (resp. $d_\nu-d_\mu$) irreducible H-submodules of dominant weight μ (resp. ν). It is clear that $\dim Z_{\mathfrak{G}}(u)\leqq\dim Z_{\mathfrak{G}'}(u)+\dim\mathscr{Z}$. Since $\dim Z_{\mathfrak{G}}(u)\geqq\dim Z_G(u)\geqq d$ trivially, it suffices to show $\dim Z_{\mathfrak{G}'}(u)+\dim\mathscr{Z}\leqq d$. We have $Z_{\mathfrak{G}'}(u)=Z_{\mathscr{V}_1}(u)\oplus Z_{\mathscr{V}_2}(u)$. Let $\mathscr{W}_1$ and $\mathscr{W}_2$ be irreducible H-modules of dominant weights μ and ν respectively. By Proposition 2 if $n_w\in H$ represents the Coxeter transformation defined earlier, $1-n_w$ is nonsingular on the zero weight spaces $(\mathscr{W}_i)_0$. By [**10**, Lemma 4.5], u is conjugate in H to an element y of the form bn_w, $b\in B$. By Proposition 3, w decomposes ${}_k\Sigma$ into $r=|{}_k\Delta|$ orbits of roots and ${}_k\Sigma_S$ into $s=|{}_k\Delta_S|$ orbits of roots, so that exactly as in [**10**, Lemma 4.3], $\dim Z_{\mathscr{W}_1}(y)\leqq r$ and $\dim Z_{\mathscr{W}_2}(y)\leqq s$. Hence, $\dim Z_{\mathscr{V}_1}(u)\leqq d_\mu r$ and $\dim Z_{\mathscr{V}_2}(u)\leqq(d_\nu-d_\mu)s$. By Proposition 2, $r=\dim(\mathscr{W}_1)_0$ and $s=\dim(\mathscr{W}_2)_0$, so $\dim Z_{\mathfrak{G}'}(u)\leqq d_\mu r+(d_\nu-d_\mu)s=\dim Z_{\mathfrak{G}'}(S)$. Since $\mathscr{Z}\subset\mathfrak{G}_0^r$ we get finally $\dim Z_{\mathfrak{G}'}(u)+\dim\mathscr{Z}\leqq\dim\mathfrak{G}_0^r=d$, as desired.

Next, assume ${}_k\Sigma$ is of type BC_r, $P\nmid 2(2r+1)$, and the subgroups K exist. Then [**6**, Proposition 5.1], there exists a simple k-split subgroup $H\supset S$ of type B_r. Put $d_\mu=\dim U_{(\mu)}$, $d_\nu=\dim U_{(\nu)}/U_{(2\nu)}$, $d_{2\nu}=\dim U_{(2\nu)}$. Define u as before with $u\in H$. Let $\rho_i: H\to GL(\mathscr{W}_i)$, $i=1, 2, 3$, be irreducible of dominant weight 2ν, μ, ν, respectively. H acts completely reducibly on the $\mathfrak{G}'$ defined as above, and $\mathfrak{G}'=\mathscr{V}_1\oplus\mathscr{V}_2\oplus\mathscr{V}_3$ where $\mathscr{V}_i$ is the direct sum of c_i copies of $\mathscr{W}_i$ ($c_1=d_{2\nu}$, $c_2=d_\mu-d_{2\nu}$, $c_3=d_\nu-d_\mu$). As before, $\dim Z_{\mathscr{W}_2}(u)\leqq r$ and $\dim Z_{\mathscr{W}_3}(u)\leqq s$, so to finish we must show $\dim Z_{\mathscr{W}_1}(u)\leqq r$. Put ${}_k\Delta=\{\alpha_1, \cdots, \alpha_r\}$ with α_i connected to α_{i+1} ($i<r$), α_1 long, α_r short. If $\lambda=\sum n_i\alpha_i\in \boldsymbol{Z}_k\Delta$, call $h(\lambda)=\sum n_i$ the height of λ. Let $(\mathscr{W}_1)_i$ be the subspace of $\mathscr{W}_1$ spanned by vectors of weight of height i, and let π_i be the projection of $\mathscr{W}_1$ onto $(\mathscr{W}_1)_i$. We say a weight α' is related to a weight λ if $\lambda=\lambda'+\alpha$ for some $\alpha\in{}_k\Delta$. If λ' is so related to $\lambda\neq 0$, since $p\neq 2$, a slight extension of the argument on p. 6 (or Proposition 2.2, [**6**]) shows $\pi_\lambda\rho_1(u)(\mathscr{W}_1)_{\lambda'}=(\mathscr{W}_1)_\lambda$, π_λ the projection of $\mathscr{W}_1$ onto $(\mathscr{W}_1)_\lambda$. Define $\beta_i=\alpha_i$ if $1\leqq i\leqq r$ and $=\alpha_{2r-i+1}$ if $r<i\leqq 2r$. For $1\leqq j\leqq 2r$, put $\gamma_{j,t}=\beta_t+\cdots+B_{j+t-1}$ for $1\leqq t\leqq\eta_j=\min\{r, r-j+2\}$, the weights of heights j. If $j>1$, $\gamma_{j-1,1}$ is related only to $\gamma_{j,1}$ and if $t>1$, $\gamma_{j-1,t}$ is related only to $\gamma_{j,t-1}$ and $\gamma_{j,t}$ (if $t\leqq\eta_j$). Hence, if $j>1$, $\pi_j\rho_1(u)$ maps $(\mathscr{W}_1)_{j-1}$ onto $(\mathscr{W}_1)_j$. Similarly, if $-1\geqq j>-2r$, $\pi_j\rho_1(u)$ maps $(\mathscr{W}_1)_{j-1}$ injectively into $(\mathscr{W}_1)_j$. The proof that $\dim(\mathscr{W}_1)_0 = r$ (Proposition 2) shows $\pi_j \rho_1(u)$ maps $(\mathscr{W}_1)_{j-1}$ injectively into $(\mathscr{W}_1)_j, j=0, 1$. Hence, $\dim Z_{\mathscr{W}_1}(u) \leqq \sum_{j \geqq 1} (\dim(\mathscr{W}_1)_j - \dim(\mathscr{W}_1)_{j+1}) = \dim(\mathscr{W}_1)_1 = r$, as desired.

Finally, we should note the characteristic zero case is much simpler, a k-regular unipotent element being obtained by taking the exponential of a k-principal nilpotent element in $\mathfrak{G}$. See [1] and [5].

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