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A NOTE ON TORSION-FREE ABELIAN GROUPS OF FINITE RANK

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ABSTRACT. Let G be a torsion-free abelian group of rank n and $X=\{x_1, \cdots, x_n\}$ a maximal set of rationally independent elements in G. It is well known that any $g \in G$ can be uniquely written $g=\alpha_1 x_1+\cdots+\alpha_n x_n$, for some $\alpha_1, \cdots, \alpha_n \in Q$, the rational numbers. This enables us to define, for any such (G, X), a collection of subgroups of Q and "natural" isomorphisms, denoted by $S(G, X)$. It is known that if G is of rank two, then G may be recovered from $S(G, X)$ is a natural way. The following result is obtained for groups of rank greater than two:

THEOREM. *Let G, G' be torsion free abelian groups of finite rank with $S(G, X)=S(G', X')$ for suitable X, X'. Let F, F' be the free subgroups of G, G' generated by X, X'. Then $G/F \cong G'/F'$.*

An additional condition is given for pairs (G, X), (G', X') such that $S(G, X)=S(G', X')$ implies $G \cong G'$.

1. **Schemes of groups.** Let G be a group[1] of rank n and $X=\{x_1, \cdots, x_n\}$ a maximal set of rationally independent elements in G. It is well known that any $g \in G$ can be uniquely written $g=\alpha_1 x_1+\cdots+\alpha_n x_n$, for some $\alpha_1, \cdots, \alpha_n \in Q$, the rational numbers. This enables us to define, for any such (G, X), a collection of subgroups of Q as follows:

DEFINITION 1. Let $1 \leqq i \leqq n$ and $i \notin J \subset \{1, 2, \cdots, n\}$. Then

$$\begin{aligned} A^i_J &= A^i_J(G, X) \\ &= \{\alpha \in Q \mid \alpha_1 x_1 + \cdots + \alpha_{i-1} x_{i-1} + \alpha x_i + \alpha_{i+1} x_{i+1} + \cdots + \alpha_n x_n \in G \\ &\qquad \text{for some set of } \alpha_j \in Q \text{ with } \alpha_j = 0 \quad \forall j \in J\}. \end{aligned}$$

The A^i_J are clearly subgroups of the additive rationals and contain the integers, Z. Furthermore, $A^i_J \supseteq A^i_{J'}$ if $J \subseteq J'$. For convenience, we write $A^i = A^i_\phi$. The A^i_J can be used to define a collection of "natural" isomorphisms.

LEMMA 1. *Given $1 \leqq i \neq j \leqq n$, with $i, j \notin J \subset \{1, 2, \cdots, n\}$, there exists an isomorphism*

$$A^i_J / A^i_{\{j\} \cup J} \overset{\theta}{\cong} A^j_J / A^j_{\{i\} \cup J}.$$

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[1] We write "group" for torsion-free abelian group throughout.

PROOF. Let $\alpha_i \in A_J^i$ and write $\sum_{k=1}^n \alpha_k x_k \in G$ for some choice of $\alpha_k \in Q$, $k \neq i$, with $\alpha_k = 0$ for $k \in J$. Then $(\alpha_i + A_{\{j\}\cup J}^i)^\theta = \alpha_j + A_{\{i\}\cup J}^j$ is the required isomorphism.

We call the collection of groups A_J^i and isomorphisms θ the scheme of (G, X), and denote it by $S(G, X)$.

LEMMA 2. *Given $1 \leqq i, j, k \leqq n$ with $i, j, k \notin J \subset \{1, \cdots, n\}$ and i, j, k distinct, then*

$$\begin{array}{ccc} A_J^i / A_{\{j\}\cup J}^i & \overset{\theta}{\cong} & A_J^j / A_{\{i\}\cup J}^j \\ \uparrow NAT & & \uparrow NAT \\ A_{\{k\}\cup J}^i / A_{\{k,j\}\cup J}^i & \overset{\theta}{\cong} & A_{\{k\}\cup J}^j / A_{\{k,i\}\cup J}^j \end{array}$$

is a commutative diagram where the θ's are the isomorphisms of Lemma 1 *and NAT is the composition of the obvious inclusion and factor maps.*

PROOF. The proof follows immediately from the definition of the maps θ.

If G has rank 2, Beaumont and Wisner [2] showed that G may be recovered from the groups A_J^i and isomorphism θ in the following way: Let x_1, x_2 be independent elements in G and $\{A^1, A_2^1, A^2, A_1^2, \theta\}$ the scheme of $(G, \{x_1, x_2\})$. Then $G' = \{(\alpha, \beta) | \alpha \in A_1, (\alpha + A_2^1)^\theta = \beta + A_1^2\}$ is isomorphic to G. Moreover, $S(G', \{(1, 0), (0, 1)\}) = S(G, \{x_1, x_2\})$.

The question then arises: To what extent does $S(G, X)$ determine G in general? The results which follow give a partial answer.

Since any group G of rank n with maximal independent set $X = \{x_1, \cdots, x_n\}$ can be identified with a subgroup of Q^n under the map $\alpha_1 x_1 + \cdots + \alpha_n x_n \to (\alpha_1, \cdots, \alpha_n)$, it suffices to consider pairs (G, X) with $Z^n \subseteq G \subseteq Q^n$ and $X = \{(1, 0, 0, \cdots, 0), (0, 1, 0, \cdots, 0), \cdots, (0, 0, \cdots, 0, 1)\}$. Let (G, X) and (G', X) be two such pairs.

LEMMA 3. *If $S(G, X) = S(G', X)$ then given any element $(\alpha_1, \alpha_2, \cdots, \alpha_n) \in G$ and fixed integers $i \neq j$, with $1 \leqq i, j \leqq n$, there exists an element $(\alpha_1', \alpha_2', \cdots, \alpha_n') \in G'$ with $\alpha_i = \alpha_i'$, $\alpha_j = \alpha_j'$. Moreover, it is possible to choose $\alpha_k' = 0$ whenever $\alpha_k = 0$.*

PROOF. By symmetry, we may take $i = 1, j = 2$. Let $K = \{k | \alpha_k = 0\}$. Since $\alpha_1 \in A_K^1$ and $A_K^1(G, X) = A_K^1(G', X)$, there exists $(\alpha_1, \beta_2, \beta_3, \cdots, \beta_n) \in G'$ with $\beta_k = 0$ for $k \in K$. Thus, $\beta_2 + A_{\{1\}\cup K}^2 = [\alpha_1 + A_{\{2\}\cup K}^1]^\theta = \alpha_2 + A_{\{1\}\cup K}^2$. Hence, $\alpha_2 - \beta_2 \in A_{\{1\}\cup K}^2$, and there exists $(0, \alpha_2 - \beta_2, \gamma_3, \gamma_4, \cdots, \gamma_n) \in G'$ with $\gamma_k = 0$ for $k \in K$. Then $(\alpha_1, \alpha_2, \alpha_3', \cdots, \alpha_n') = (0, \alpha_2 - \beta_2, \gamma_3, \cdots, \gamma_n) + (\alpha_1, \beta_2, \cdots, \beta_n) \in G'$ with $\alpha_k' = 0$ for $k \in K$.

LEMMA 4. *Let G, G', X be as in Lemma 3, and p a prime integer. Let $g=(\alpha_1, \cdots, \alpha_n) \in G$ and $g'_1, \cdots, g'_t \in G'$ be such that $\bar{g}=g+Z^n$, $\bar{g}'_1=g'_1+Z^n, \cdots, \bar{g}'_t=g'_t+Z^n$ are independent elements in the p-primary component of Q^n/Z^n. Then there exists $g'=(\alpha'_1, \cdots, \alpha'_n) \in G'$, with* order$(\bar{g}') \geqq$ order$(\bar{g})=p^k$, *such that $\bar{g}'_1, \cdots, \bar{g}'_t$, $\bar{g}'$ are independent. If $\alpha_r=a_r/p^k$, $\alpha_s=a_s/p^{k'}$, for integers a_r, a_s with $(a_r, p)=(a_s, p)=1$, $k' \leqq k$, then g' can be chosen so that $c\alpha_r=\alpha'_r$, $c\alpha_s=\alpha'_s$, where c is an integer with $(c, p)=1$. Furthermore, we can choose $\alpha'_i=0$ whenever $\alpha_i=0$.*

PROOF. We proceed by induction on the number, m, of nonzero coordinates of g. If $m=1, 2$, then $g \in G'$ by Lemma 3 and the result follows. Suppose now that $g=(\alpha_1, \cdots, \alpha_m, 0, 0, \cdots, 0)$. We assume, without loss of generality, that all integral components of g are zero, and, by renumbering if necessary, that the nonzero coordinates appear first. Write $\alpha_i=a_i/p^{k_i}$, $1 \leqq k_i \leqq k$, where $(a_i, p)=1$. Again by renumbering, assume $r=1$, $s=2$. Thus, $k_1=k$. By Lemma 3, there exists

$$h_1 = (a_1/p^{k_1}, a_2/p^{k_2}, \beta_3, \cdots, \beta_m, 0, \cdots, 0) \in G'.$$

Choose an integer c with $(c, p)=1$ such that

$$h_2 \equiv ch_1 = (ca_1/p^{k_1}, ca_2/p^{k_2}, b_3/p^{l_3}, \cdots, b_m/p^{l_m}, 0, \cdots, 0).$$

Note that $\{\bar{g}'_1, \cdots, \bar{g}'_t, c\bar{g}\}$ is still independent. (The integer c times a power of p is divisible by the denominators of $\beta_3 \cdots \beta_m$.)

Case I. Suppose $l_i=k_i$ and $p|ca_i-b_i$ for $3 \leqq i \leqq m$. Then order$(\bar{h}_2)=p^k$. We show $\{\bar{g}'_1, \cdots, \bar{g}'_t, \bar{h}_2\}$ is independent. It suffices to show $\{p^{j_1-1}\bar{g}'_1, \cdots, p^{j_t-1}\bar{g}'_t, p^{k-1}\bar{h}_2\}$ is independent in $\bar{G}'[p]$, where order $\bar{g}'_i=p^{j_i}$. Suppose $u_1p^{j_1-1}\bar{g}'_1+\cdots+u_tp^{j_t-1}\bar{g}'_t+up^{k-1}\bar{h}_2=0$ with $(p, u)=1$. Clearly, $u_1p^{j_1-1}\bar{g}'_1+\cdots+u_tp^{j_t-1}\bar{g}'_t+up^{k-1}c\bar{g}\neq 0$. Subtraction yields $up^{k-1}(c\bar{g}-\bar{h}_2)\neq 0$. But, since $p|ca_i-b_i$, $i=3, 4, \cdots, m$, $up^{k-1}(c\bar{g}-\bar{h}_2)=0$, a contradiction. Thus, in Case I, $g'=h_2$ is the desired element of G'.

Case II. Assume without loss of generality that $l_3 \neq k_3$, or that $p \nmid ca_3-b_3$. It is then easy to show that $1/p^{k_3} \in A^3_{\{1, m+1, m+2, \cdots, n\}}$. Thus there exist $\Gamma_1=(0, x, ca_3/p^{k_3}, x_4, \cdots, x_m, 0, 0, \cdots, 0) \in G$ and (by Lemma 3) $\Gamma'_1=(0, x, ca_3/p^{k_3}, x'_4, \cdots, x'_m, 0, \cdots, 0) \in G'$. Applying Lemma 3 again to $\Gamma_2=cg-\Gamma_1$, we have

$$\Gamma'_2 = (ca_1/p^{k_1}, ca_2/p^{k_2} - x, 0, y'_4, \cdots, y'_m, 0, 0, \cdots, 0) \in G'.$$

Consider $h_2-\Gamma'_1-\Gamma'_2=(0, 0, b_3/p^{l_3}-ca_3/p^{k_3}, z'_4, \cdots, z'_m, 0, 0, \cdots, 0)$. Since $l_3 \neq k_3$ or $p \nmid b_3-ca_3$, it follows that $1/p^{k_3} \in A^3_{\{1, 2, m+1, m+2, \cdots, n\}}$. Let

$$\Gamma_3 = (0, 0, ca_3/p^{k_3}, v_4, \cdots, v_m, 0, 0, \cdots, 0) \in G$$

and

$$\Gamma'_3 = (0, 0, ca_3/p^{k_3}, v'_4, \cdots, v'_m, 0, 0, \cdots, 0) \in G'.$$

Finally, let

$$\Gamma_4' = \Gamma_1' + \Gamma_2' - \Gamma_3' = (ca_1/p^{k_1}, ca_2/p^{k_2}, 0, w_4', \cdots, w_m', 0, 0, \cdots, 0).$$

If $\{\bar{g}_1', \cdots, \bar{g}_t', \bar{\Gamma}_4'\}$ is independent, it is easy to construct the desired element $g' \in G'$. If $\{\bar{g}_1', \cdots, \bar{g}_t', \overline{cg-\Gamma_3}\}$ is independent, the result follows from the induction hypothesis, after multiplication by a suitable integer prime to p.

Otherwise, $\bar{\Gamma}_4'$ is dependent on, and $\bar{\Gamma}_3$ is independent of the set $\{\bar{g}_1', \cdots, \bar{g}_t'\}$. Multiplying Γ_3 by a suitable integer d, prime to p, and applying the induction hypothesis, we obtain

$$\Gamma_3'' = (0, 0, cda_3/p^{k_3}, v_4'', \cdots, v_m'', 0, 0, \cdots, 0) \in G'$$

independent of $\bar{g}_1', \cdots, \bar{g}_t'$, Now $\bar{\Gamma}_3''+d\bar{\Gamma}_4'$ is still independent, and we can choose e prime to p such that $g'=e(\Gamma_3''+d\Gamma_4')$ is the desired element.

LEMMA 5. *Let G, G', X be as in Lemma 3. Let $\bar{g}_1, \cdots, \bar{g}_t$ be independent elements in the p-primary component of $\bar{G}$. Then there exist $\bar{g}_1', \cdots, \bar{g}_t'$ independent in the p-primary component of $\bar{G}'$ with* $\operatorname{order}(\bar{g}_i') \geqq \operatorname{order}(\bar{g}_i)$.

PROOF. The result follows by using induction on t and applying Lemma 4. We now prove the main theorem.

THEOREM A. *Let G, G' be groups of rank n with maximal independent sets Y, Y' respectively, such that $S(G, Y)=S(G', Y')$. Let F, F' be the free subgroups of G, G' generated by Y, Y' respectively. Then $G/F \cong G'/F'$.*

PROOF. As before, we identify G and G' with subgroups of Q^n containing Z^n, such that Y and Y' are identified with the set X of standard basis vectors. Write

$$G/Z^n = \sum \oplus A_p, \qquad G'/Z^n = \sum \oplus A_p'$$

where A_p and A_p' are the respective p-primary components. Note that A_p and A_p' are subgroups of $(Q^n/Z^n)_p$, and therefore

$$A_p = \sum_{i=1}^{r} \oplus Z(p^{k_i}),$$

$$A_p' = \sum_{i=1}^{r'} \oplus Z(p^{k_i'}) \qquad \text{where } 1 \leqq k_i \leqq \infty, \quad 1 \leqq k_i' \leqq \infty.$$

Applying Lemma 5, we have that $A_p \cong A_p'$ for all p.

Note that the converse to Theorem A is false. It is easy to construct nonisomorphic subgroups G and G' of $Q \oplus Q$ which contain $Z \oplus Z$ with $G/Z \oplus Z \cong G'/Z \oplus Z$. Since groups of rank two are determined up to isomorphism by their schemes, $S(G, X) \neq S(G', X')$ for any X, X'.

2. **The intersection property.** In this section we give a condition under which a group may be recovered from its scheme, $S(G, X)$.

DEFINITION 2. The groups A_J^i associated with (G, X) are said to satisfy the intersection property iff $A_J^i \cap A_{J'}^i = A_{J\cup J'}^i$ for all $1 \leqq i \leqq n$ and $i \notin J \cup J'$, where J and J' are subsets of $\{1, \cdots, n\}$. In this case we will also say (G, X) satisfies the intersection property.

REMARKS. It is immediate that if G has rank $\leqq 2$, (G, X) satisfies the intersection property for any maximal independent set $X \subseteq G$. If (G_i, X_i) has the intersection property for $i = 1, 2, \cdots, n$, then $(\sum_{i=1}^n \oplus G_i, \bigcup_{i=1}^n X_i')$ has the intersection property, where X_i' denotes the natural embedding of X_i into the direct sum. Let $v = \sum_{k=0}^\infty v_k p^k$, $w = \sum_{k=0}^\infty w_k p^k$ be p-adic units such that $1, v, w$ are rationally independent. Let $V_0 = W_0 = 0$, $V_i = \sum_{k=0}^{i-1} v_k p^k$, $W_i = \sum_{k=0}^{i-1} w_k p^k$, for $i > 1$. It can be shown using the methods of [**1**] that

$$G = \{(r/p^k, r/p^k V_k + t, r/p^k W_k + l) \mid (r, p) = 1, t, l \in Z\}$$

is a strongly indecomposable rank 3 group; and it is easy to verify directly that (G, X) has the intersection property with $X = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

THEOREM B. *Let (G, X) satisfy the intersection property for G a group of rank n and $X = \{x_1, \cdots, x_n\}$ a maximal independent subset. Let $S(G, X) = \{A_J^i, \theta\}$. Then*

$$\bar{G} = \{(\alpha_1, \cdots, \alpha_n) \in Q^n \mid \alpha_i \in A^i, (\alpha_i + A_j^i)^\theta = \alpha_j + A_i^j, i, j = 1, \cdots, n\}$$

is isomorphic to G under the map $(\alpha_1, \cdots, \alpha_n) \to \sum_{i=1}^n \alpha_i x_i$.

PROOF. We show that $(\alpha_1, \cdots, \alpha_n) \in \bar{G}$ iff $\sum_{i=1}^n \alpha_i x_i \in G$. Clearly, by definition of the maps θ, $\sum_{i=1}^n \alpha_i x_i \in G$ implies $(\alpha_1, \cdots, \alpha_n) \in \bar{G}$. Let $(\alpha_1, \cdots, \alpha_n) \in \bar{G}$. Then $\alpha_1 \in A^1$, so $g_1 = \alpha_1 x_1 + \alpha_2' x_2 + \cdots + \alpha_n' x_n \in G$ for some $\alpha_2', \cdots, \alpha_n'$. But $\alpha_2' + A_1^2 = (\alpha_1 + A_2^1)^\theta = \alpha_2 + A_1^2$, so $\alpha_2 - \alpha_2' \in A_1^2$. Thus, there exists $g_2 = (\alpha_2 - \alpha_2')x_2 + \alpha_3'' x_3 + \cdots + \alpha_n'' x_n \in G$. Now $g_1 + g_2 = \alpha_1 x_1 + \alpha_2 x_2 + (\alpha_3' + \alpha_3'')x_3 + \cdots + (\alpha_n' + \alpha_n'')x_n \in G$. This yields $\alpha_3 - (\alpha_3' + \alpha_3'') \in A_1^3 \cap A_2^3$. Since (G, X) satisfies the intersection property, we have $\alpha_3 - (\alpha_3' + \alpha_3'') \in A_{12}^3$, and there exists $g_3 = [\alpha_3 - (\alpha_3' + \alpha_3'')]x_3 + \alpha_4''' x_4 + \cdots + \alpha_n''' x_n \in G$. We continue in this way, obtaining $g_1, \cdots, g_n$, such that $\sum_{i=1}^n \alpha_i x_i = g_1 + g_2 + \cdots + g_n \in G$.

COROLLARY. *Let (G, X) satisfy the intersection property, and let $S(G, X) = S(G', X')$. Then $G \cong G'$.*

REMARK (ADDED IN PROOF). Theorem B and the above Corollary do not hold in general. There exist two nonisomorphic groups of rank 3 which have the same schemes with respect to the standard basis.

REFERENCES

1. R. A. Beaumont and R. S. Pierce, *Torsion-free rings*, Illinois J. Math. **5** (1961), 61–98. MR **26** #6212.

2. R. A. Beaumont and R. J. Wisner, *Rings with additive group which is a torsion-free group of rank two*, Acta Sci. Math. (Szeged) **20** (1959), 105–116. MR **21** #5651.

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