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ON BAZILEVIČ FUNCTIONS

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ABSTRACT. The authors present a short proof of the well-known result that the Bazilevič functions of type α, α positive, are univalent. Moreover, those functions are "relatives" of the close-to-convex functions.

The purpose of this note is to give a short and direct proof of the following well-known and often proved result [1], [3], [4].

THEOREM. *The set $B(\alpha)$ of analytic functions $f(z)$, defined in the unit disc Δ by the relation*

$$f(z) = \left[\alpha \int_0^z \frac{\sigma^\alpha(\zeta)}{\zeta} p(\zeta)\, d\zeta\right]^{1/\alpha} = z + \cdots \tag{1}$$

where $\sigma(z)=z+\cdots$ is a fixed starlike univalent function in Δ, $p(z)=1+\cdots$ is a fixed Carathéodory function and α is a positive constant, consists of univalent functions. Moreover, if $\alpha=m/n$ (rational), then $[f(z^n)]^{m/n}$ is a close-to-convex m-valent function.

PROOF. From (1) we obtain the interesting and provocative relation

$$zf'(z)/f^{1-\alpha}(z)\sigma^\alpha(z) = p(z), \tag{2}$$

discovered by Pommerenke [3].

First we consider the case that α is rational, $\alpha=m/n$. If we set $g(z)=[f(z^n)]^{m/n}$ or $f(z^n)=[g(z)]^{n/m}$ in (2), then we find

$$\frac{z^n f'(z^n)}{f^{1-m/n}(z^n)\sigma^{m/n}(z^n)} = p(z^n) = \frac{zg'(z)}{m\sigma^{m/n}(z^n)} \equiv \frac{g'(z)}{\Phi'(z)}, \tag{3}$$

where

$$\Phi(z) \equiv m \int_0^z \frac{\sigma^{m/n}(z^n)}{z}\, dz = z^m + \cdots$$

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is a convex m-valent function defined in the unit disc Δ. We now apply (3) and a result due to Umezawa [5] to conclude that $\Phi(z)$ is a close-to-convex m-valent function. Hence $[f(z^n)]^{1/n}$ and $f(z)$ are univalent in Δ.

If α is not rational, then it can be approximated by a sequence of positive rationals $\{\Omega_k\}$. Each Ω_k gives rise to a Bazilevič function $f_k(z)$, defined by (1) with α replaced by Ω_k. A "normal families" argument, making use of the fact that the functions $\sigma(z)$ and $p(z)$ remain fixed in (1), (2) and (3), shows that the $f_k(z)$ converge uniformly on compact subsets of Δ to the $f(z)$ defined by (1). Since each $f_k(z)$ is univalent in Δ, and since $f'(0)=1$, it follows that $f(z)$ is univalent in Δ. This completes our proof.

REMARK 1. The set $B(\alpha)$ is the closure (under the usual topology) of the set $[f(z) | f\in B(m/n), m, n=1, 2, \cdots]$. Hence each $f\in B(\alpha)$ is a limit of functions "related" to the close-to-convex univalent functions.

REMARK 2. The set $B(\alpha)$ considered here is a (relatively small) subset of the set B actually introduced by Bazilevič. The proofs of the univalence of each member of B given by Bazilevič, Pommerenke and Sheil-Small are all different and more sophisticated than the one we have presented here.

REMARK 3. The device we used in our proof had been used by Keogh and Miller, but for another purpose [2].

ADDED IN PROOF. Recent papers by Takatsuka [Duke Math. J. **33** (1966), 583–593 and Trans. Amer. Math. Soc. **120** (1965), 72–82] and Sakaguchi [J. Math. Soc. Japan **14** (1962), 312–321] have results that impinge on the one contained in this note.

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