L²-SOLUTIONS TO \( y'' + c(t)y' + a(t)b(y) = 0 \)

ALLAN KROOPNICK

Abstract. Two theorems are presented giving sufficient conditions for all solutions to \( y'' + c(t)y' + a(t)b(y) = 0 \) to be bounded and square integrable over the nonnegative real line.

In this paper, sufficient conditions are given for all solutions to the differential equation

\[ y'' + c(t)y' + a(t)b(y) = 0 \]

to be square integrable over \([0, \infty)\). In proving this, we shall show that the derivatives are also square integrable over the same interval. We now state and prove our first result.

\textbf{Theorem I.} Suppose \( c(t), a(t) \) and \( b(y) \) are all continuous functions. Furthermore, let \( c(t) > a > 0, \ a(t) > a, \) and \( a'(0) \leq 0 \) on \([0, \infty), \) and

\[ \lim_{|y| \to \infty} b(y) = \int_0^\infty b(u) \, du = \infty, \]

then all solutions to (1) are bounded as \( t \to \infty \) and \( \int_0^\infty y'^2 \, dt < \infty. \)

\textbf{Proof.} Multiply (1) by \( 2y' \) to get

\[ 2y'y'' + 2c(t)y'^2 + 2a(t)b(y)y' = 0. \]

Integrating by parts from 0 to \( t \) we have

\[ y'(t)^2 + 2 \int_0^t c(s)y''(s) \, ds + 2a(t)B(y(t)) - 2 \int_0^t a'(s)B(y(s)) \, ds = K \]

where \( K = y'(0)^2 + 2a(0)B(y(0)). \) The above implies \( |y| \) remains bounded as \( t \to \infty \) and \( \int_0^\infty y'^2 \, dt < \infty. \) Otherwise, the left side would become infinite which is impossible.

We now give sufficient conditions for the solutions to be \( L^2 \)-solutions.

\textbf{Theorem II.} The hypotheses are the same as Theorem I. In addition, if \( c'(t) \leq 0 \) on \([0, \infty)\) and \( yb(y) \geq \beta y^2 \) where \( \beta > 0, \) then \( \int_0^\infty y^2 \, dt < \infty. \)

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Proof. Multiply (1) by \( y \) to get

\[
(4) \quad yy'' + c(t)yy' + a(t)b(y)y = 0.
\]

Integrating by parts from 0 to \( t \) we have

\[
(5) \quad y(t)y'(t) - \int_0^t y'(s)^2 \, ds + \frac{y^2(t)}{2} c(t)
\]

\[
- \frac{1}{2} \int_0^t y^2(s) \frac{dc}{ds} \, ds + \int_0^t a(s)b(y)y \, ds = y(0)y'(0) + \frac{y^2(0)}{2} c(0).
\]

Letting \( t \to \infty \), we immediately conclude

\[
(6) \quad \int_0^\infty y^2 \, dt < \infty.
\]