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COUNTING p-SUBGROUPS[1]

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ABSTRACT. There are many theorems which state that the number of p-subgroups of a finite group, where these p-subgroups satisfy varying conditions, is congruent 1 modulo p. We derive here a simple theorem which has all these special theorems as corollaries.

1. **Introduction.** G stands for a finite group of order g and p for a prime number dividing g. The theorem, mentioned in the abstract, is:

THEOREM. *Let K be a subgroup of G of order p^m. If $m \leqq n$ and p^n divides g, the number of subgroups of G of order p^n which contain K is congruent* 1 *modulo p.*

Several special cases of the theorem occur in the literature. For example, if $m=0$, one obtains that the number of subgroups of G of order p^n is congruent 1 modulo p; or, if n is chosen as large as possible, one obtains that the number of Sylow p-subgroups of G which contain K is congruent 1 modulo p [**1**, p. 152].

2. **The proof of the theorem.** The theorem is trivial when $n=m$.

Case 1. $n=m+1$. Suppose that K is contained in t subgroups $H_1, \cdots, H_t$ of G of order p^{m+1}. Since p^n divides g, $t \geqq 1$, and it is well known that K is normal in each H_i. Consequently, the groups H_i are subgroups of the normalizer N of K in G, whence t is the number of subgroups of order p of N/K. Clearly, $p \mid [N:K]$ and there are of course several elementary ways of showing that the number of subgroups of order p of a group whose order is divisible by p is congruent 1 modulo p.

Case 2. $n=m+1+s$, where $s \geqq 1$. We make the induction hypothesis that the theorem has been proved for $n=m, m+1, \cdots, m+s$. Let $L_1, \cdots, L_t$ be the subgroups of G of order p^{m+s} which contain K, and $H_1, \cdots, H_u$ the subgroups of order p^{m+s+1} containing K. Since $p^n \mid g$, t and u are positive, and we consider the $t \times u$ matrix (a_{ij}) where $a_{ij}=1$ if $L_i \subset H_j$ and 0 otherwise. The row sum $\sum_{j=1}^{u} a_{ij}$ of the matrix is denoted

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by r_i, and the column sum $\sum_{i=1}^{t} a_{ij}$ by c_j, which gives the following tableau

$$\begin{array}{c|ccc|c} & H_1 & \cdots & H_u & \\ L_1 & & & & r_1 \\ \cdot & & & & \cdot \\ \cdot & & a_{ij} & & \cdot \\ \cdot & & & & \cdot \\ L_t & & & & r_t \\ & c_1 & \cdots & c_u & \end{array}$$

By the induction hypothesis, $t \equiv 1 \pmod p$ and we must show that $u \equiv 1 \pmod p$. Since $r_1 + \cdots + r_t = c_1 + \cdots + c_u$, it is sufficient to show that each $r_i \equiv 1 \pmod p$ and each $c_j \equiv 1 \pmod p$. However, r_i is the number of subgroups of order p^{m+s+1} of G which contain L_i, whence $r_i \equiv 1 \pmod p$ by Case 1. Furthermore, c_j is the number of subgroups of order p^{m+s} of H_j which contain K, whence $c_j \equiv 1 \pmod p$ by the induction hypothesis. Done.

3. **Normal p-subgroups.** The theorem also enables one to obtain the standard statements which refer to normal p-subgroups. The proof of the following corollary shows how this works.

COROLLARY. *Let everything be as in the theorem, but assume furthermore that G is a p-group and K is normal in G. Then, the number of normal subgroups of G of order p^n which contain K is congruent* 1 *modulo p* [**1**, p. 129].

PROOF. Let $\{H_1, \cdots, H_t\}$ be the set of subgroups of G of order p^n which contain K. Since K is normal in G, G acts on this set of groups by inner automorphisms. Since G is a p-group, the number of fixed points of G in the permutation representation $(G, \{H_1, \cdots, H_t\})$ is congruent $t \pmod p$. However, G leaves H_i fixed if and only if H_i is normal in G, and $t \equiv 1 \pmod p$ by the theorem.

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