ADDITIVITY AND INDEFINITE INTEGRATION FOR McSHANE'S $P$-INTEGRAL

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Abstract. Suppose $[a, b]$ is a closed real number interval, $A = \{(p, q); (p, q) \subset (a, b)\}$, and $U$ is a real valued function on $[a, b] \times A$. For $c \in (a, b)$, necessary and sufficient conditions are given for $P$-integrability of $U$ on $(a, c]$ and $(c, b]$ to imply $P$-integrability on $(a, b]$. Suppose $U$ is $P$-integrable on $(a, b]$ and $g(x) = P \int_{a}^{x} U$ for each $x \in [a, b]$. Necessary and sufficient conditions are given for $g$ to be respectively continuous, bounded, and of bounded variation.

1. Preliminaries. Suppose $a = t_0 < t_1 < \cdots < t_n = b$ and $x_1, x_2, \ldots, x_n \in [a, b]$. Then $\Pi = \{(x_1, (t_0, t_1)), \ldots, (x_n, (t_{n-1}, t_n))\}$ is called a $P$-partition of $[a, b]$. If $\delta$ is a positive real valued function on $[a, b]$, then the statement that $\Pi$ is $\delta$-fine means that $(t_i-t_{i-1}, t_i)$ is $(x_i-\delta(x_i), x_i+\delta(x_i))$ for $i = 1, \ldots, n$.

The statement that $U$ is $P$-integrable on $(a, b]$ means that there is a real number denoted by $P \int_{a}^{b} U$ such that if $\epsilon > 0$ there is a positive real valued function $\delta$ on $[a, b]$ such that if $\Pi = \{(x_i, (t_{i-1}, t_i))\}_{i=1}^{n}$ is any $\delta$-fine $P$-partition of $(a, b]$ then

$$\left| \sum_{i=1}^{n} U(x_i, (t_{i-1}, t_i)) - P \int_{a}^{b} U \right| < \epsilon.$$

Variations in the following lemma can be found in ([1], [2], [3]).

Lemma K. If $U$ is $P$-integrable on $(a, b]$ and $\epsilon > 0$, there is a positive real valued function $\delta$ on $[a, b]$ such that if $\Pi = \{(x_i, (t_{i-1}, t_i))\}_{i=1}^{n}$ is any $\delta$-fine $P$-partition of $(a, b]$ then

$$\sum_{i=1}^{n} \left| U(x_i, (t_{i-1}, t_i)) - P \int_{t_{i-1}}^{t_i} U \right| < \epsilon.$$

2. Additivity. E. J. McShane [1] proved that if $U$ is $P$-integrable on $(a, b]$ and $c \in (a, b)$ then $U$ is $P$-integrable on $(a, c]$ and $(c, b]$ and that
The following theorem gives necessary and sufficient conditions for a form of the converse to hold.

**Theorem 2.1.** Suppose $c \in (a, b)$ and $U$ is $P$-integrable on $(a, c]$ and $(c, b]$. Then each of the following statements is equivalent to each of the others:

(i) $U$ is integrable on $(a, b]$,

(ii) if $\varepsilon > 0$, there is a positive number $\gamma$ such that if $\sup\{a, c - \gamma\} < x < c < y < \inf\{b, c + \gamma\}$, then $|U(c, (x, c]) + U(c, (c, y]) - U(c, (x, y])] < \varepsilon$, and

(iii) if $\varepsilon > 0$ there is a positive number $\lambda$ such that if $\sup\{a, c - \lambda\} < x < c < y < \inf\{b, c + \lambda\}$,

then $|P \int_a^x U + P \int_c^y U - U(c, (x, y])] < \varepsilon$.

**Proof.** Suppose (i) is true and $\varepsilon > 0$. By Lemma K there is a positive function $\delta$ on $[a, b]$ satisfying Lemma K for $\varepsilon/3$. Then if $\sup\{a, c - \delta(c)\} < x < c < y < \inf\{b, c + \delta(c)\}$,

\[
\begin{align*}
\left| P \int_x^c U - U(c, (x, c]) \right| &< \varepsilon/3, \\
\left| P \int_c^e U - U(c, (c, y]) \right| &< \varepsilon/3, \quad \text{and} \\
\left| P \int_x^e U - U(c, (x, y]) \right| &< \varepsilon/3.
\end{align*}
\]

Combining these inequalities yields

$|U(c, (x, c]) + U(c, (c, y]) - U(c, (x, y])] < \varepsilon$.

Thus (i)$\Rightarrow$(ii). Also, (3) yields

$\left| P \int_x^c U + P \int_c^e U - U(c, (x, y]) \right| < \varepsilon/3 < \varepsilon$.

Thus (i)$\Rightarrow$(iii).

Assume (ii) is true, let $\varepsilon > 0$, and let $\gamma$ be the positive number from (ii) for $\varepsilon/3$. There exist positive functions $\delta_1$ and $\delta_2$ corresponding to the intervals $(a, c]$ and $(c, b]$, which satisfy Lemma K for $\varepsilon/3$. Let

$\delta(x) = \inf\{\delta_1(x), c - x\}$ if $x \in [a, c),$

$= \inf\{\delta_1(c), \delta_2(c), \gamma\}$ if $x = c$, and

$= \inf\{\delta_2(x), x - c\}$ if $x \in (c, b]$. 

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Suppose \( \Pi = \{ (t_i, (x_{i-1}, x_i)) \}_{i=1}^{n} \) is a \( \delta \)-fine \( P \)-partition of \( (a, b] \). If \( c = x_j \) for some \( j \),

\[
\left| \sum_{i=1}^{n} U(t_i, (x_{i-1}, x_i)) - P \int_{a}^{c} U - P \int_{c}^{b} U \right|
\leq \left| \sum_{i=1}^{n} U(t_i(x_{i-1}, x_i)) - P \int_{a}^{c} U \right| + \left| \sum_{i=1}^{n} U(t_i, (x_{i-1}, x_i)) - P \int_{c}^{b} U \right|
< \varepsilon/3 + \varepsilon/3 < \varepsilon.
\]

If \( c \in (x_j-1, x_j) \) for some \( j \),

\[
\left| \sum_{i=1}^{n} U(t_i, (x_{i-1}, x_i)) - P \int_{a}^{c} U - P \int_{c}^{b} U \right|
\leq \left| \sum_{i=1}^{n-1} U(t_i, (x_{i-1}, x_i)) - P \int_{x_{i-1}}^{x_i} U \right| + \left| U(c, (x_{j-1}, c)) - P \int_{x_{j-1}}^{c} U \right|
+ \left| U(c, (x_{j-1}, c)) + U(c, (c, x_j)) - U(c, (x_{j-1}, x_j)) \right|
< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.
\]

Thus \( (ii) \Rightarrow (i) \).

Suppose \( (iii) \) is true and \( \varepsilon > 0 \). Let \( \lambda \) be the positive number from \( (iii) \) for \( \varepsilon/3 \). Let \( \delta_1 \) and \( \delta_2 \) be positive functions on \( (a, c] \) and \( (c, b] \) respectively which satisfy Lemma K for \( \varepsilon/3 \). Choose \( \gamma = \inf \{ \delta_1(c), \delta_2(c), \lambda \} \). Then if \( \sup \{ a, c - \gamma \} < x < c < y < \inf \{ b, c + \gamma \} \),

\[
\left| U(c, (x, c)) + U(c, (c, y)) - U(c, (x, y)) \right|
\leq \left| U(c, (x, c)) - P \int_{x}^{c} U \right| + \left| U(c, (c, y)) - P \int_{c}^{y} U \right|
+ \left| P \int_{x}^{c} U + P \int_{c}^{y} U - U(c, (x, y)) \right|
< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.
\]

Thus \( (iii) \Rightarrow (ii) \).

3. Indefinite integration. Suppose \( U \) is \( P \)-integrable on \( (a, b] \) and \( g(x) = P \int_{a}^{x} U \) for each \( x \in [a, b] \). \( g \) is called the indefinite integral of \( U \) on \( [a, b] \).

**Theorem 3.1.** \( g \) is bounded on \( [a, b] \) if and only if there is a positive function \( \delta \) on \( [a, b] \) and a number \( M \) such that if \( (c, d) \subset (a, b] \), \( t \in [a, b] \), and \( (c, d) \subset (t - \delta(t), t + \delta(t)) \), then \( |U(t, (c, d))| < M \).
Proof of Necessity. Suppose \( g \) is bounded and let \( L \) denote a bound of \( |g| \). There is a positive function \( \delta \) satisfying Lemma K for \( \varepsilon = 1 \).

Suppose \((c, d) \subseteq (a, b)\), \( t \in [a, b] \) and \((c, d) \subseteq (t-\delta(t), t+\delta(t))\). Then there is a \( \delta \)-fine \( P \)-partition of \((a, b)\) with \((t, (c, d))\) as a member. Thus 
\[
|U(t, (c, d)) - P \int_t^e U| < 1 \quad \text{and} \quad |U(t, (c, d))| < 1 + |P \int_t^e U| \leq 1 + |g(d) - g(c)| \leq 1 + 2L.
\]

Proof of Sufficiency. Suppose \( U, \delta, \) and \( M \) satisfy the hypothesis and assume \( g \) is not bounded. Let \( \{t_i\} \) denote an increasing sequence of numbers in \([a, b]\) such that \( |g(t_i)| > i \) for each \( i \) and let \( t \in (a, b) \) denote the limit of \( \{t_i\} \). If no such sequence exists, there would be a similar decreasing sequence and a similar argument would follow. By Lemma K, there exists a positive function \( \delta' \) on \([a, t]\) such that if 
\[
\Pi = \{(t_i, (x_{i-1}, x_i))\}_{i=1}^n
\]
is a \( \delta' \)-fine \( P \)-partition of \((a, t]\), then 
\[
\sum_{i=1}^n |U(t_i, (x_{i-1}, x_i)) - P \int_{x_{i-1}}^{x_i} U| < 1.
\]

Let \( \gamma = \inf\{\delta(t), \delta'(t)\} \) and \( j \) a positive integer such that \( 0 < |t_j - t| < \gamma \) and 
\[
|g(t_j) - g(t)| > M + 1.
\]
Since there is a \( \delta' \)-fine \( P \)-partition of \((a, t]\) with \((t, (t_j, t])\) an element,
\[
|U(t, (t_j, t]) - \int_{t_j}^t U| < 1,
\]
thus \( |g(t) - g(t_j)| - 1 < |U(t, (t_j, t])| \quad \text{and} \quad M < |U(t_j, (t_j, t])| \) which contradicts the hypothesis.

Theorem 3.2. \( g \) is continuous on \([a, b]\) if and only if for each \( \varepsilon > 0 \) there is a positive function \( \delta \) on \([a, b]\) such that if \((c, d) \subseteq (a, b)\), \( t \in [a, b] \) and \((c, d) \subseteq (t-\delta(t), t+\delta(t))\), then 
\[
|U(t, (c, d))| < \varepsilon.
\]

Proof of Necessity. Suppose \( g \) is on \([a, b]\) and \( \varepsilon > 0 \). Let \( \delta' \) be positive real from the definition of uniform continuity for \( \varepsilon/2 \) and \( \delta'' \) be the positive function from Lemma K for \( \varepsilon/2 \). Let \( \delta(t) = \inf\{\delta', \delta''(t)\} \). Then if \( t \in [a, b] \) and \((p, q) \subseteq (a, b)\) such that \((p, q) \subseteq (t-\delta(t), t+\delta(t))\), by Lemma K 
\[
|U(t, (p, q)) - P \int_p^q U| < \varepsilon/2,
\]
\[
|U(t, (p, q)]) < \varepsilon/2 + |g(q) - g(p)|,
\]
and since \( |p - q| < \delta, |U(t, (p, q)]) < \varepsilon \).

Proof of Sufficiency. Suppose \( U \) satisfies the hypothesis and \( \varepsilon > 0 \).
Let $\delta$ be the positive function from the hypothesis for $\varepsilon/2$ and $\delta^*$ be the positive function from Lemma K for $\varepsilon/2$. Let $\delta = \inf\{\delta', \delta^*\}$. Then if $x, y \in [a, b]$ and $(x, y) \subseteq (x-\delta(x), x+\delta(x))$,

$$
|g(x) - g(y)| = \left| P \int_x^y U \right| 
\leq \left| P \int_x^y U - U(x, (x, y)) \right| + |U(x, (x, y))| 
< \varepsilon/2 + \varepsilon/2 = \varepsilon.
$$

Thus $g$ is continuous.

**Theorem 3.3.** $g$ is of bounded variation on $[a, b]$ if and only if $|U|$ is $P$-integrable on $(a, b)$. Furthermore, if $g$ is of bounded variation on $[a, b]$, then the total variation of $g$ on $[a, b]$ is $\int_a^b |U|$.

**Proof of Necessity.** Suppose $g$ is of bounded variation and let $Vg$ denote the total variation of $g$ on $[a, b]$ and let $\varepsilon > 0$. There is a partition $a = x_0 < x_1 < \cdots < x_n = b$ such that $0 \leq Vg - \sum_{i=1}^n |g(x_i) - g(x_{i-1})| < \varepsilon/2$.

Let $\delta'(t) = \inf\{x_i - t, t - x_{i-1}\}$ if $t \in (x_{i-1}, x_i)$,

$$
= \frac{1}{2} \inf\{x_i - x_{i-1}, x_{i+1} - x_i\} \text{ if } t = x_i \text{ for } i = 1, \cdots, n - 1,
$$

$$
= \frac{1}{2}(x_n - x_{n-1}) \text{ if } t = x_n,
$$

$$
= \frac{1}{2}(x_1 - x_0) \text{ if } t = x_0.
$$

Let $\delta^*$ be the positive function from Lemma K for $\varepsilon/2$ and $\delta = \inf\{\delta', \delta^*\}$.

Suppose $\Pi = \{(t_i, (y_{i-1}, y_i))\}_{i=1}^m$ is a $\delta$-fine $P$-partition of $(a, b)$. Then $a = y_0 < y_1 < \cdots < y_m = b$ is a refinement of $a = x_0 < x_1 < \cdots < x_n = b$ and thus $0 \leq Vg - \sum_{i=1}^m |g(y_i) - g(y_{i-1})| < \varepsilon/2$ and $0 \leq Vg - \sum_{i=1}^m |P \int_{y_{i-1}}^{y_i} U| < \varepsilon/2$. It follows from Lemma K that

$$
\left| \sum_{i=1}^m |U(t_i, (y_{i-1}, y_i))| - \sum_{i=1}^m P \int_{y_{i-1}}^{y_i} U \right| < \frac{\varepsilon}{2}.
$$

Combining inequalities yields

$$
\left| \sum_{i=1}^m |U(t_i, (y_{i-1}, y_i))| - Vg \right| < \varepsilon.
$$

Therefore $|U|$ is $P$-integrable and $P \int_a^b |U| = Vg$.

**Proof of Sufficiency.** Suppose $|U|$ is $P$-integrable and $\varepsilon > 0$. Let $\delta^*$ be a positive function which satisfies Lemma K for both $U$ and $|U|$ for $\varepsilon/2$. 

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Let \( a = x_0 < x_1 < \cdots < x_n = b \) be any partition of \([a, b]\). Define \( \delta \) as in the necessary proof and let \( \Pi = \{(t_i, (y_{i-1}, y_i))\}_{i=1}^{m} \) be any \( \delta \)-fine \( P \)-partition of \((a, b)\).

Lemma K yields
\[
\sum_{i=1}^{m} \left| U(t_i, (y_{i-1}, y_i)) - P \int_{y_{i-1}}^{y_i} U \right| < \frac{\varepsilon}{2}
\]
and thus
\[
\sum_{i=1}^{m} \left| U(t_i, (y_{i-1}, y_i)) - |g(y_i) - g(y_{i-1})| \right| < \frac{\varepsilon}{2}.
\]

Also, Lemma K gives
\[
\sum_{i=1}^{m} \left| U(t_i, (y_{i-1}, y_i)) - P \int_{y_{i-1}}^{y_i} |U| \right| < \frac{\varepsilon}{2}.
\]

Combining inequalities yields
\[
\left| \sum_{i=1}^{m} |g(y_i) - g(y_{i-1})| - P \int_{a}^{b} |U| \right| < \varepsilon.
\]

Therefore it follows from the definition of bounded variation that \( g \) is of bounded variation and that \( Vg \) is \( P \int_{a}^{b} |U| \).

REFERENCES


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