MULTIPLIERS FOR THE SPACE OF ALMOST-CONVERGENT FUNCTIONS ON A SEMIGROUP

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Abstract. Let $S$ be a countably infinite left amenable cancellative semigroup, $FL(S)$ the space of left almost-convergent functions on $S$. The purpose of this paper is to show that the following two statements concerning a bounded real function $f$ on $S$ are equivalent: (i) $f \cdot FL(S) \subset FL(S)$; (ii) there is a constant $\alpha$ such that for each $\varepsilon > 0$ there exists a set $A \subset S$ satisfying (a) $\varphi(X_A) = 0$ for each left invariant mean $\varphi$ on $S$ and (b) $|f(x) - \alpha| < \varepsilon$ if $x \in S \setminus A$.

1. Let $S$ be a semigroup, $m(S)$ the space of bounded real functions on $S$ with the sup norm. $\varphi \in m(S)^*$ is called a left invariant mean on $S$ if $\|\varphi\| = 1$, $\varphi \geq 0$ and $\varphi(l_s f) = \varphi(f)$ for $s \in S$ and $f \in m(S)$, where $l_s f \in m(S)$ is defined by $(l_s f)(t) = f(st)$, $t \in S$. The set of left invariant means on $S$ is denoted by $ML(S)$. If $ML(S)$ is nonempty, then $S$ is said to be left amenable [2]. A bounded real function $f$ on a left amenable semigroup is called left almost-convergent if $\varphi(f)$ equals a fixed constant $d(f)$ as $\varphi$ runs through $ML(S)$. The set of all left almost-convergent functions, denoted by $FL(S)$, is a vector subspace of $m(S)$ and it contains constant functions. But, in general, it is not closed under multiplication. The purpose of this paper is to study this aspect of $FL(S)$ and our main result is the following.

Theorem. Let $S$ be a countable left-cancellative left amenable semigroup without finite left ideals. Then the following two statements concerning a function $f \in m(S)$ are equivalent:

(i) $f$ is a multiplier of $FL(S)$, i.e., $f \cdot FL(S) \subset FL(S)$;

(ii) $f$ is $S$-convergent to a constant $\alpha$, i.e., for a given $\varepsilon > 0$ there exists a set $A \subset S$ such that

(a) $\varphi(X_A) = 0$ for each $\varphi \in ML(S)$, and

(b) $|f(x) - \alpha| < \varepsilon$ if $x \in S \setminus A$.

2. From now on $S$ will always denote a left-cancellative left amenable semigroup without finite left ideals.

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Remarks. (1) A set $A \subseteq S$ is said to be left almost-convergent if its characteristic function $\chi_A$ is left almost-convergent. In this case we denote $d(\chi_A)$ by $d(A)$. Roughly speaking, a set $A \subseteq S$ is left almost-convergent if it is evenly distributed in $S$ with respect to the semigroup structure and $d(A)$ indicates the density of $A \subseteq S$. In particular, $d(A) = 0$ means that $A$ is sparsely distributed in $S$. (The set $A$ in the statement (ii) of the above Theorem is such a set.) For example, when $S = \mathbb{N}$, the additive semigroup of positive integers, a set $A \subseteq \mathbb{N}$ is left almost-convergent if and only if
\[
\lim_{n \to \infty} \frac{1}{n} \text{Card}\{k, k + 1, \ldots, k + n - 1\} \cap A = d(A)
\]
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\]
exists uniformly in $k$ [7].

(2) Since $S$ contains no finite left ideals, $d(B) = 0$ for each finite subset $B \subseteq S$ (cf. [5]). Therefore if $f$ converges to $a$ at infinity, i.e., if given $\varepsilon > 0$ there exists a finite set $B \subseteq S$ such that $|f(x) - a| < \varepsilon$ whenever $x \notin B$, then $f$ $S$-converges to $a$. On the other hand, the space of $S$-convergent functions is much smaller than $FL(S)$. Indeed, $FL(S)$ separates points of $\beta S$ [6], the Stone-Cech compactification of the discrete set $S$, while it is easy to see that $f \in m(S)$ is $S$-convergent if and only if $f$ is a constant on the set

\[
K(S) = \text{cl}_{\beta S} \cup \{\text{supp } \phi : \phi \in ML(S)\},
\]
(cf. [1]). Here we consider a bounded real function on $S$ as a continuous function on $\beta S$ and a mean on $m(S)$ as a probability measure on $\beta S$. In particular, if $\phi \in ML(S)$, supp $\phi$ denotes the support of the measure $\phi$.

(3) As in [2], let $EG$ denote the smallest class of groups which contains all finite groups, all abelian groups and is closed under the following four ways of constructing new groups from given ones: (a) subgroup; (b) factor group; (c) group extension; and (d) direct limits.

Each group in $EG$ is amenable and they constitute all the known amenable groups [2]. If we assume that $S$ is an infinite group in $EG$, then a stronger result is known [1]: $f \in m(S)$ is $S$-convergent if and only if

\[
\begin{align*}
(a) & f \cdot \chi_A \in FL(S) \text{ for each left almost-convergent set } A, \\
(b) & f^n \in FL(S), n = 1, 2, \ldots ;
\end{align*}
\]
in particular, if $A \subseteq S$ and $A \cap B$ is left almost-convergent for each $\chi_B \in FL(S)$, then $d(A) = 1$ or $0$. It is not clear whether our Theorem yields the same conclusion. The proof of our Theorem is completely different from the proof in [1].

3. Proof of the Theorem. (ii) $\Rightarrow$ (i) is easy, cf. [1].

(i) $\Rightarrow$ (ii). Let $f$ be a multiplier of $FL(S)$. To show that $f$ is $S$-convergent, it suffices to show that $f \equiv d(f)$ on supp $\phi$. We claim that this follows from the following assertion:

\[
\begin{align*}
(a) & \text{ If } g \in FL(S) \text{ and } \phi \in ML(S), \text{ then } \phi(fg) = \phi(f)\phi(g).
\end{align*}
\]
Indeed, if (α) holds and if φ ∈ ML(S) then

φ((f − d(f))^2) = (φ(f) − d(f))^2 = 0.

Therefore f ≡ d(f) on supp φ as we wanted.

Proof of (α). We shall consider l^1(S)^* = m(S) and l^1(S)^** = m(S)^*.

If φ ∈ l^1(S) and h ∈ m(S), then h(φ) = h(φ) = ∑_{t ∈ S} h(t)φ(t). Since S is left amenable and countable there exists a sequence φ_n in l^1(S) such that \|φ_n\|_1 = 1, φ_n ≥ 0, and \lim_n (φ_n(h) − φ_n(λh)) = 0 for each h ∈ m(S) and each \lambda ∈ S [5, Lemma 5.1]. We shall need the following two well-known facts (cf. [3, §9]):

(β) If g ∈ FL(S), then \lim_n φ_n(g) = d(g).

(γ) FL(S) = the closed linear span of \{l_x h : x ∈ S, h ∈ m(S)\} \cup \{χ_S\}.

Let \lambda ∈ S be fixed. Set \psi_n = φ_n · f − l_x φ_n · f, i.e., \psi_n(t) = \phi_n(t) − φ_n(x t) f(x t), t ∈ S. Then \psi_n ∈ l^1(S). We claim that \psi_n is a weak Cauchy sequence in l^1(S). Indeed, if h ∈ m(S),

\[\psi_n(l_x h) = \sum_{t ∈ S} (\phi_n(t) f(t)(x t) − \phi_n(x t) f(x t) h(x t)) = \sum_{t ∈ S} \phi_n(t) f(t)(x t) h(x t) + \sum_{t ∈ S \setminus x S} \phi_n(t) f(t) h(t) = \psi_n(f · (l_x h) − h) + \psi_n(f h · \chi_{S \setminus x S}) = a_n + b_n.\]

Note that f · (l_x h − h) ∈ FL(S), since f is a multiplier of FL(S) and l_x h − h ∈ FL(S). Therefore by (β) \lim_n a_n = d(f · (l_x h − h)). Note also that \chi_{S \setminus t S} = χ_{l_x S} = χ_{S} = 1, i.e., \psi_n(\chi_{S \setminus t S}) = 0 for each \lambda ∈ S. Therefore \chi_{S \setminus t S} is left almost-convergent to zero. By (β) again, we get

\[|b_n| \leq \|f\|_∞ \|h\|_∞ \phi_n(\chi_{S \setminus t S}) \to 0 \text{ as } n \to ∞.\]

Therefore we have obtained:

(δ) \lim_n \phi_n(l_x h) = d(f · (l_x h − h)), h ∈ m(S).

Since S is left-cancellative, each k ∈ m(S) is of the form l_x h for some h ∈ m(S). Therefore \lim_n \psi_n(h) exists for each h ∈ m(S), i.e., \psi_n is a weak Cauchy sequence as we claimed. Since l^1(S) is weakly sequentially complete [4, p. 374], there exists ψ ∈ l^1(S) such that ψ = lim_n \psi_n in the weak topology. Certainly, \psi(t) = lim_n \psi_n(t) for \lambda ∈ S. On the other hand,

\[
\lim_n \psi_n(t) = lim_n \phi_n(\chi_{t \setminus t}) = 0,
\]

since \chi_{t \setminus t} ∈ FL(S) and d(\chi_{t \setminus t}) = 0 (cf. Remark (1)). Hence

\[\psi(t) = \lim_n (\phi_n(t) f(t) − \phi_n(x t) f(x t)) = 0.\]

So, \psi ≡ 0. By (δ), d(f · (l_x h − h)) = 0 = d(f) · d(l_x h − h). It is of course true
that \( d(f \cdot c_{\Phi}) = d(f) \cdot d(c_{\Phi}) \). Hence (\( \alpha \)) follows from (\( \gamma \)) and the above observation.

REFERENCES


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