$C^k$, WEAKLY HOLOMORPHIC FUNCTIONS ON
ANALYTIC SETS1

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Abstract. Let $V$ be a complex analytic set and $p \in V$. Let $\mathcal{O}(V)$, $\mathcal{O}(V)$, and $C^k(V)$ denote respectively the rings of germs of holomorphic, weakly holomorphic, and $k$-times continuously differentiable functions on $V$. Spallek proved that there exists sufficiently large $k$ such that $C^k(V) \cap \mathcal{O}(V) = \mathcal{O}(V)$. In this paper I give a new proof of this result for curves and hypersurfaces which also establishes that the conduction number of the singularity is an upper bound for $k$. This estimate also holds for any pure dimensional variety off of a subvariety of codimension two.

Let $V$ be a complex analytic set and $p \in V$. Let $\mathcal{O}(V)$, $\mathcal{O}(V)$, and $C^k(V)$ denote respectively the rings of germs of holomorphic, weakly holomorphic, and $k$-times continuously differentiable functions on $V$. In [6], Spallek proved that there exists sufficiently large $k$ such that $C^k(V) \cap \mathcal{O}(V) = \mathcal{O}(V)$. In this paper I give a new proof of this result for curves and hypersurfaces which also establishes that the conduction number of the singularity is an upper bound for $k$. This estimate also holds for any pure dimensional variety off of a subvariety of codimension two.

An element $u \in \mathcal{O}$ is said to be a universal denominator if $u\mathcal{O} \subset \mathcal{O}$. Let $I$ be the ideal of $\mathcal{O}$ of all functions vanishing on $\text{Sing}(V)$ and $J$ be the ideal of universal denominators. Then $\text{locus}(J) \subset \text{Sing} V$ [3, p. 56], so by the Hilbert Nullstellensatz there is a positive integer $N$ such that $I^N \subset J$. The main result of this paper is that $k$ can be chosen so that $k \leq N$.

Siu has proven [5] that if $k(p)$ is the minimal value of $k$ such that $C^k \cap \mathcal{O} = \mathcal{O}$ at the point $p \in V$, then the function $k(p)$ is bounded on compact subsets of $V$. This result also follows from the above estimate, by the coherence of the ideal sheafs of $I$ and $J$. (The ideal sheaf of $J$ is coherent [2, Theorem 22] because it is the kernel of $\mathcal{O} \rightarrow \text{Hom}_\mathcal{O}(\mathcal{O}, \mathcal{O}/\mathcal{O})$.)

This estimate is, in general, not the best possible. In an earlier work [1], for the example of a curve in $C^2$ normalized by a map $t \rightarrow (t^p, t^qu(t))$ where $u(t)$ is a unit, $p > q$, and $p$ and $q$ are relatively prime, it was shown that

Received by the editors June 30, 1972.


Key words and phrases. $C^k$, weakly holomorphic, conduction number.

1 Research supported by National Science Foundation SD GU3171.

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$k = [(p/q)(q-2)] + 1$ and $N = [(p/q)(q-1)]$, where $[x]$ for any real number $x$ is the greatest integer less than or equal to $x$.

The original estimate for $k$ obtained by Spallek [6] seems a bit obscure in the case of a nonisolated singularity. This is made clearer by Siu in [5] and Spallek in [8]:

Suppose $A$ is an analytic set of pure dimension $r$, $\pi: A \rightarrow \mathbb{C}^r$ a branched covering of sheeting order $\mu$, $z_{r+1}$ a direction in $\mathbb{C}^n$ which separates the fibers of $\pi$ almost everywhere, and $\delta$ the discriminant of the minimal polynomial in $\mathcal{O}$ for $z_{r+1}$ over $\mathcal{O}$; then $k \leq \mu(m+1)$ where $\delta(\text{locus}(\delta))^m \subseteq \delta_\mathcal{O}$. Now $m$ is related to the conduction number $N$, but not necessarily equal—depending upon whether the projection $\pi$ has minimal multiplicity—so the estimate in this paper is better approximately by a factor of the minimal multiplicity. For the above mentioned case of a curve in $\mathbb{C}^2$, direct computation shows that Spallek's estimate is $k \leq p(q-1) + q$.

I am greatly indebted to Professor John Stutz for correcting an error in an earlier version of this paper.

1. Suppose $V$ is a complex analytic hypersurface in $\mathbb{C}^n$, the projection $\pi: \mathbb{C}^n \rightarrow \mathbb{C}^{n-1}$ to the first $n-1$ coordinates gives a $q$ sheeted branched cover of $V$ with branch set $B$, $B' = \pi(B)$ and $z' = \pi(z)$. Now $\pi$ induces a homomorphism $\pi^*\mathcal{O} \rightarrow \mathcal{O}/\mathcal{I}(V) = \mathcal{O}(V)$ making $\mathcal{O}(V)$ into a finitely generated $\pi^*\mathcal{O}$ module with generators $1, z, \ldots, z_{n-1}$. Hence if $f \in \mathcal{O}(V)$, then $f$ can be written as $\sum_{i=0}^{n-1} b_i(z') z_{n-1}^{q-i-1}$. For any weakly holomorphic function $f$, there is a canonical attempted extension to the ambient space, which is in fact holomorphic, if $f$ is holomorphic: for $z' \notin B'$, let

$$g(z', z_n) = \sum_{i=1}^{n-1} \left( \prod_{k \neq j} (z_{n} - \alpha_k(z')) \right) f(z', \alpha_j(z'))$$

$$= \sum_{i=0}^{n-1} (-1)^i b_i(z') z^{q-i},$$

$$b_i(z') = \sum_{j=1}^{n-1} \sigma_i(\alpha_1(z'), \ldots, \alpha_i(z'), \ldots, \alpha_q(z')) \left( \prod_{k \neq j} (z_{n} - \alpha_k(z')) \right) f(z', \alpha_j(z'))$$

where hatted terms are deleted, $\sigma_i$ is the elementary symmetric polynomial of degree $i$, and $\{\alpha_j(z'): 1 \leq j \leq q\}$ are the values of $z_n$ on the fiber $\pi^{-1}(z')$.

For $z \in V$, $z_n = \text{some } \alpha_j$, so $g(z) \equiv f(z)$. The coefficients $b_i$ are well defined (do not depend upon the ordering of the $\alpha_i$'s), $b_i \in \mathcal{O}(\mathbb{C}^r - B')$ and $g \in \mathcal{O}(\mathbb{C}^n - B' \times \mathbb{C}^{n-r})$. By the Riemann removable singularities theorem, $g$ extends holomorphically to $\mathbb{C}^n$ if and only if the $b_i$ are bounded near $B'$. (If $g$ is holomorphic, then so is $(\partial^{q-1}/\partial z_n)g = (q-1)! b_0(z')$, etc.)
If \( f \in \mathcal{C}(V) \), then as pointed out by Spallek [5, Abschnitt 6], the Newton Interpolation Formula [4, pp. 10–16] says that if

\[
[f_1, \ldots, f_q] = \sum_{j=1}^{q} \frac{f(z', \alpha_j(z'))}{\prod_{j \neq k} (\alpha_j(z') - \alpha_k(z'))},
\]

then there exists a complex constant \( \lambda \) with \( |\lambda| \leq 1 \) and real numbers \( \delta_1, \ldots, \delta_q \geq 0 \) with \( \sum \delta_i = 1 \) such that

\[
[f_1, \ldots, f_q] = (\lambda/(q - 1)!)(\partial^{q-1}/\partial z_n)f(z', \delta_1 \alpha_1 + \cdots + \delta_q \alpha_q).
\]

Now

\[
\sigma_i(\alpha_1, \ldots, \alpha_q) = \sigma_i(\alpha_1, \ldots, \alpha_q) - \sigma_{i-1}(\alpha_1, \ldots, \alpha_q)
\]

\[
= \sum_{l=0}^{i} (-1)^l \sigma_{l-i}(\alpha_1, \ldots, \alpha_q)
\]

so

\[
b_i(z') = \sum_{l=0}^{i} (-1)^l \sigma_{l-i}(\alpha_1, \ldots, \alpha_q) [(z_n f)_1, \ldots, (z_n f)_q]
\]

and it follows immediately that \( b_i \) is bounded near \( B' \) by the continuity of \( (\partial^{q-1}/\partial z_n)f \).

2. First we consider the one-dimensional case. Let \( V \) be normalized by a map \( \theta(t) = (t^q, t^{p_1} u_1(t), \ldots, t^{p_{n-1}} u_{n-1}(t)) \) where each \( p_i \geq q \) and each \( u_i \) is a holomorphic function with \( u_i(0) \neq 0 \). Let \( f \in C^k(V) \cap \mathcal{C}(V) \), and \( T_0^k(f) \) be the \( k \)th order Taylor series of \( f \) about the origin; write \( T_0^k(f) = P_k f + Q_k f \) where \( P_k f \) is a homomorphic polynomial and \( Q_k f \) contains the antiholomorphic terms. It is a standard fact that \( f - T_0^k f = o(|z|^k) \). However even more is true:

**Lemma.** \( (f-P_k f) \theta(t) = o(\theta(t)^k) = o(t^k) \).

This is Lemma 3 of [1] and is also essentially contained in [5, paragraph 2.2].

Let \( h = f - P_k f \); we have that \( h \) is also weakly holomorphic, \( h \) is holomorphic if and only if \( f \) is holomorphic, \( h \) is precisely as differentiable as \( f \) and that \( h(\theta(t)) = o(t^k) \) since \( P_k h \equiv 0 \). Hence \( h/z_k^N \) is weakly holomorphic. Since \( z_1^N \) is a universal denominator, \( h z_1^{N-k} \) is holomorphic; for \( k = N \) we have that \( h \) is holomorphic. Thus \( C^N(V) \cap \mathcal{C}(V) = \mathcal{C}(V) \).

More generally, for a variety \( V \) of pure dim \( r \) in \( C^n \), let

\[
C = \text{Sing} \ (\text{Sing} \ V)
\]

\[\cup \{ p \in V | \dim C_4(V, p) > r \} \cup \{ p \in V | \dim C_6(V, p) > r + 1 \}\]
where $C_4(V, p)$ and $C_5(V, p)$ are the fourth and fifth Whitney tangent cones to $V$ at $p$ [10]. Then $C$ is an analytic subset of $V$ of codimension at least two [9, Proposition 3.6] and every $p \in V - C$ has an open neighborhood so that after a local biholomorphic change of coordinates the following hold:

(i) For each irreducible component $V_i$ of $V$, $V_i \cap \text{Sing } V = \text{Sing } V_i = C^{r-1}$ [9, Proposition 2.10, 2.12, and 4.5].

(ii) Each component has a one-to-one nonsingular normalization [9, Proposition 4.2] $\phi: D \to V_i$ given by

$$\phi(t_1, \ldots, t_r) = (t_1, \ldots, t_{r-1}, t_r^q, \phi_{r+1}(t), \ldots, \phi_n(t)),$$

where $q$ is the sheeting order of $\pi|_{V_i}$ and $\pi(x_1, \ldots, x_n) = (x_1, \ldots, x_r)$. The branching set of this projection is just $\phi(t_r = 0) = C^{r-1}$.

Now let $\text{Cond}_p(V)$ denote the conduction number of the variety at the point $p$ as defined in the introduction. If $V_i$ is a component of $V$ it is clear that any universal denominator for $V$ is a universal denominator for $V_i$ and since $\text{Sing } V_i = \text{Sing } V$, we have that $\text{Cond}_p(V) \geq \text{Cond}_p(V_i)$.

For any fixed $s = (t_1, \ldots, t_{r-1})$ consider the curve $W_s$ in $V_i$ given by $t_r = \phi(s, t_r)$. Since this curve $W_s$ lies in $s \times C^{n-r+1}$, weakly holomorphic functions on $W_s$ extend to weakly holomorphic functions on $V_i$ by ignoring the first $r-1$ variables. Hence any universal denominator for $V_i$ is a universal denominator for $W_s$ and $\text{Cond}_p(V_i) \geq \text{Cond}_p(W_s)$.

Suppose $f \in C^k \cap \mathfrak{m}$, $k \geq \text{Cond}_p(V)$, and $r = n-1$; recall the canonical extension of §1, $\sum b_i(z')z'^{r-1}$, with $b_i \in \mathcal{O}(C^{r-C^{r-1}})$. Since $W_s$ is a hypersurface in $s \times C^2$ and $k \geq \text{Cond}_p(\phi(s, 0)(W_s))$ for each $s$, by the one-dimensional case we have that $b_i$ is bounded on each line $L_s = \{(s, z_r): z_r \in C\}$.

We need to conclude that $b_i$ extends holomorphically to $C^r$. To do this consider the Laurent power series expansion of $b_i$ in $C^{r-C^{r-1}}$:

$$b_i(z') = \sum_{m=-\infty}^{+\infty} a_m(z_1, \ldots, z_{r-1})z_r^m, \quad a_m \in \mathcal{O}(C^{r-1}),$$

$$a_m(z') = \frac{1}{2\pi i} \oint_{\xi = \gamma} b_i(z'', \xi) \frac{d\xi}{\xi^{m+1}}.$$

Choose an $s \in C^{r-1}$ such that for each $a_m$ which is not identically zero, $a_m(s) \neq 0$; if there are any negative exponents of $z_r$ in the above power series expansion, $b_i$ is not bounded on the line $L_s$ (neither a pole nor an essential singularity is bounded).

So far we have shown that $k \leq \text{Cond}_p(V)$ for $p \in V - C$. By coherence of the ideal sheafs of $I(\text{Sing } V)$ and $J$, if $c \in C$, then $\text{Cond}_c(V) \geq \text{Cond}_p(V)$.
for all $p$ near $c$. If $f \in C^k \cap \mathcal{O}$ with $k \geq \text{Cond}_c(V)$ then by the last paragraph $b_i \in \mathcal{O}(C^r - \pi(C))$; but dim $\pi(C) \leq r - 2$ so by Hartog's theorem [3, p. 59], $b_i \in \mathcal{O}(C^r)$ and $f \in \mathcal{O}(V)$.

3. Even without the assumption about the codimension of $V$, it follows that for pure $r$-dimensional $V$ there exists an analytic subset $W$ of $V$ with dim $W \leq r - 2$ such that for every $p \in V - W$, $k(p) \leq N(p)$. Of course this implies the result of the last section since for algebraic complete intersections, singularities in codim two are removable [11].

Instead of directly exhibiting the holomorphic extension, we must resort to more delicate results in sheaf theory, due to Spallek [8, Satz 3.2].

If $f$ is a weakly holomorphic function on $V$, $1 \leq q \leq r$, and $f$ restricted to each $q$-dimensional parallel section is holomorphic, then there is an analytic subset $B^q$ of $V$ (not depending upon $f$) of dimension at most $r - q - 1$ so that $f$ is holomorphic on $V - B^q$.

It was shown in the previous section that by restricting to $V - C$, we have $f$ holomorphic on each $s \times C^{n-r}$ and $q = \dim V \cap (s \times C^{n-r}) = 1$ so dim $B^1 \leq r - 2$ and $f$ is holomorphic on $V - (C \cup B^1)$.

BIBLIOGRAPHY


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