PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 39, Number 1, June 1973

ON A UNILATERAL PROBLEM ASSOCIATED WITH ELLIPTIC OPERATORS[1]

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ABSTRACT. Let $\mathscr{A}$ be a uniformly elliptic linear differential expression of second order, defined on the bounded domain $\Omega \subset \boldsymbol{R}^m$, and let $\beta \subset \boldsymbol{R} \times \boldsymbol{R}$ be a maximal monotone graph. Under some growth assumption on β it is shown that for any given $f \in L^2(\Omega)$ the problem: $\mathscr{A}u + \beta(u) \ni f$ on Ω, $u=0$ on $\partial\Omega$, admits a strong solution. It is not required that $\mathscr{A}$ is monotone.

1. **Introduction.** Let $\Omega \subset \boldsymbol{R}^m$ ($m \geqq 1$) be an open bounded set with smooth boundary $\partial\Omega$. Let further

$$\mathscr{A} = \sum_{|\alpha|,|\beta| \leqq 1} (-1)^{|\alpha|} D^\alpha a_{\alpha\beta} D^\beta,$$

where

$$a_{\alpha\beta} \in C^{|\alpha|}(\bar{\Omega}) \qquad (|\alpha|, |\beta| \leqq 1),$$

$$\sum_{|\alpha|=|\beta|=1} a_{\alpha\beta}\xi_\alpha\xi_\beta \geqq c\,|\xi|^2, \qquad c > 0, \quad x \in \Omega, \quad \forall \xi \in \boldsymbol{R}^m,$$

be a uniformly elliptic differential expression of second order (all our spaces are assumed to be real). Moreover let $\beta \subset \boldsymbol{R} \times \boldsymbol{R}$ be a maximal monotone graph, i.e. a mapping of $\boldsymbol{R}$ to $2^{\boldsymbol{R}}$ such that (i) $\forall t_1 \in \beta(s_1)$, $t_2 \in \beta(s_2)$, the inequality $(t_1 - t_2)(s_1 - s_2) \geqq 0$ holds, and (ii) there exists no monotone graph in $\boldsymbol{R} \times \boldsymbol{R}$ extending β properly. Let $D(\beta) = \{s \in \boldsymbol{R}: \beta(s) \neq \varnothing\}$. Then for each $s \in D(\beta)$, $\beta(s)$ is a closed interval. Suppose $0 \in \beta(0)$.

We prove in the present paper that for any $f \in L^2(\Omega)$ the boundary value problem

$$(*) \qquad \begin{aligned} \mathscr{A}u + \beta(u) &\ni f \quad \text{on } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Received by the editors July 11, 1972.

AMS (MOS) subject classifications (1970). Primary 35J60, 47H05; Secondary 46E35.

Key words and phrases. Uniformly elliptic linear differential operator, maximal monotone mapping, Yosida approximation, homotopy arguments, Sobolev space, strong solution.

[1] Research supported by the Schweizerischer Nationalfonds.

admits a *strong* solution u, provided

$$(1) \qquad |\beta^0(s)| \geqq \psi(s)\,|s| \quad \text{with} \lim_{|s|\to\infty} \psi(s) = +\infty.$$

Here

$$\begin{aligned} |\beta^0(s)| &= \operatorname*{Min}_{t\in\beta(s)} |t|, \quad && s \in D(\beta), \\ &= +\infty, && \text{otherwise,} \end{aligned}$$

and $\beta^0(s)s \geqq 0$ ($s \in D(\beta)$).

EXAMPLE. Let

$$\begin{aligned} \beta(s) &= (-\infty, 0], \qquad && s = 0, \\ &= se^s, && s > 0. \end{aligned}$$

Then we solve the following problem:

$$\begin{aligned} &u \geqq 0 \quad \text{on } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega; \\ &u(x) > 0 : \mathscr{A}u + ue^u = f, \\ &u(x) = 0 : \qquad \mathscr{A}u \geqq f. \end{aligned}$$

If $\mathscr{A}$ consists only of its "principal" part $\mathscr{A}' = \sum_{|\alpha|=|\beta|=1} (-1)^{|\alpha|} D^\alpha a_{\alpha\beta} D^\beta$, with possibly a term of order 0 with nonnegative coefficient added, the strong solvability of (∗) follows (without the additional assumption (1)) from the theory of maximal monotone operators (e.g. Brézis-Crandall-Pazy [**3**, Example 1]). We reduce our more general case to the monotone situation by employing a homotopy argument.

2. **Statement of the result.** Let $A : L^2(\Omega) \supset D(A) \to L^2(\Omega)$ be the linear operator given by

$$(2) \qquad D(A) = H_0^1(\Omega) \cap H^2(\Omega), \qquad Au = \mathscr{A}u, \qquad u \in D(A).$$

The maximal monotone graph $\beta \subset \boldsymbol{R} \times \boldsymbol{R}$ extends uniquely to a (maximal monotone) operator $\bar\beta : L^2(\Omega) \to 2^{L^2(\Omega)}$ by

$$(3) \qquad v \in \bar\beta(u) \Leftrightarrow v(x) \in \beta(u(x)) \quad \text{a.e. on } \Omega.$$

THEOREM. *Let A and $\bar\beta$ be the operators defined in* (2) *and* (3), *respectively, and suppose* (1) *holds. Then for any given $f \in L^2(\Omega)$ there exists $u \in L^2(\Omega)$ with $Au + \bar\beta(u) \ni f$; i.e., $u \in H_0^1(\Omega) \cap H^2(\Omega)$, and for some $b \in L^2(\Omega)$ we have*

$$\begin{aligned} &b(x) \in \beta(u(x)) && \text{a.e. on } \Omega, \\ &(\mathscr{A}u)(x) + b(x) = f(x) && \text{a.e. on } \Omega. \end{aligned}$$

REMARKS. 1. One can show that $A + \bar\beta : L^2(\Omega) \to 2^{L^2(\Omega)}$ is a generalized pseudo-monotone operator in the sense of Browder-Hess [**5**]. This mapping is in general however neither densely defined nor quasi-bounded, so that the abstract existence theorems of **[5]** do not apply.

2. It might seem simpler to study problem $(*)$ with operators acting from $H_0^1(\Omega)$ to $2^{H^{-1}(\Omega)}$. But besides loosing regularity of the solutions, one encounters serious interpretational difficulties concerning the (maximal monotone) operator $\tilde{\beta}:H_0^1(\Omega)\to 2^{H^{-1}(\Omega)}$ derived from $\bar{\beta}$ in an obvious way.

3. For the weak solvability of somewhat related problems see the author's notes **[8]**–**[10]**.

3. **A solvability criterion for functional equations.** Let X be a real reflexive Banach space. We denote by X^* its conjugate space, and by (f, u) the value of $f\in X^*$ at $u \in X$. A mapping T from X to X^* is called of type $(S)^+$ provided for any sequence $\{u_n\}$ in X converging weakly to some u, for which $\limsup(Tu_n, u_n-u)\leqq 0$, the strong convergence $u_n\to u$ follows ("$\to$" and "$\rightharpoonup$" denote strong and weak convergence, respectively). The operator T is further demicontinuous if $v_n\to v$ implies $Tv_n\rightharpoonup Tv$. For a subset G of X, cl(G) means its closure, ∂G its boundary.

PROPOSITION. *Let X be a real reflexive Banach space, G an open bounded subset of X containing* 0, *and $T_tu=T(u, t)$ a mapping of $X\times[0, 1]$ into X^* having the following properties*:

(i) *For fixed t, T_t is demicontinuous and of type* $(S)^+$;

(ii) *T_tu is continuous in t, uniformly with respect to $u\in$* cl(G);

(iii) $(T_0u, u)\geqq 0$, $\forall u\in\partial G$.

Suppose that $T_tu\neq 0$, $\forall u\in\partial G$, $\forall 0\leqq t\leqq 1$. Then the equation $T_1u=0$ is solvable in G.

The proof resembles that of Theorem 1 in Hess **[7]**; we do not reproduce it here.

4. **Some remarks on maximal monotone operators.** Let $\beta\subset \boldsymbol{R}\times\boldsymbol{R}$ be a maximal monotone graph with $0\in\beta(0)$. Then the mapping $I+\lambda\beta$ is surjective for each $\lambda>0$, and $(I+\lambda\beta)^{-1}$ is a contraction in $\boldsymbol{R}$. The Yosida approximation β_λ $(\lambda>0)$ of β is defined by $\beta_\lambda(s)=\lambda^{-1}(I-(I+\lambda\beta)^{-1})(s)$; it is a monotone increasing, Lipschitz continuous function defined on $\boldsymbol{R}$. Clearly $0=\beta_\lambda(0)$ for each $\lambda>0$.

Let $\bar{\beta}:L^2(\Omega)\to 2_{L^2(\Omega)}$ and $\overline{\beta_\lambda}:L^2(\Omega)\to L^2(\Omega)$ denote the extensions of β and β_λ, respectively, as defined in (3). One proves easily that $\bar{\beta}$ and $\overline{\beta_\lambda}$ are maxima monotone mappings, with $\overline{\beta_\lambda}$ being Lipschitz continuous. Moreover $\overline{\beta_\lambda}$ is the Yosida approximation of $\bar{\beta}$: $\overline{\beta_\lambda}=(\bar{\beta})_\lambda$. Since $0=\overline{\beta_\lambda}(0)$ for $\lambda>0$, we have $(\overline{\beta_\lambda}(u), u)_{L^2(\Omega)}\geqq 0$ for all $u\in L^2(\Omega)$.

For an extensive treatment of maximal monotone operators in reflexive Banach spaces we refer to Browder **[4]**. Yosida approximations to m-accretive operators are discussed in Kato **[11]** (cf. also Crandall-Pazy **[6]**), to maximal monotone mappings in Brézis-Crandall-Pazy [**3**] and Browder-Hess [**5**].

5. **Proof of the Theorem.** We set $H=L^2(\Omega)$ and $X=H_0^1(\Omega)$. By identification of the Hilbert space H with its conjugate space, $X\subset H\subset X^*$. Let $(\cdot,\cdot)$ denote either the inner product in H, or the duality pairing between X^* and X. We distinguish the norms in the various spaces by a subscript. Let $\mathscr{A}'=-\sum_{|\alpha|=|\beta|=1} D^\alpha a_{\alpha\beta} D^\beta$ be the "principal" part of $\mathscr{A}$ and A' the operator in H induced by $\mathscr{A}'$ on the domain $D(A')=X\cap H^2(\Omega)$. Then

$$(4)\qquad \|A'u\|_H \geqq c_1 \|u\|_{H^2(\Omega)}, \qquad c_1>0, \qquad \forall u\in D(A')$$

(e.g. Agmon [**1**, Theorem 9.8]). Let further $\mathscr{A}''=\mathscr{A}-\mathscr{A}'$ and A'' be the mapping in H induced by $\mathscr{A}''$, with $D(A'')=X$.

(i) Let $f\in H$ be given. We claim that there exists $R>0$ such that

$$(5)\qquad \begin{gathered} 0\notin A'u+\bar{\beta}(u)+t(A''u-f), \\ \forall u\in D(A')\cap D(\bar{\beta}) \ \text{ with } \|u\|_X=R, \qquad \forall 0\leqq t\leqq 1. \end{gathered}$$

Suppose to the contrary that to each n we find $u_n\in D(A')\cap D(\bar{\beta})$ with $\|u_n\|_X=n$ and $0\leqq t_n\leqq 1$ such that $0\in A'u_n+\bar{\beta}(u_n)+t_n(A''u_n-f)$. Let $b_n\in\bar{\beta}(u_n)$ be the element with

$$(6)\qquad 0=A'u_n+b_n+t_n(A''u_n-f).$$

Taking the inner product of (6) with b_n, we obtain

$$(7)\qquad (A'u_n,b_n)+\|b_n\|_H^2=-t_n(A''u_n-f,b_n).$$

It is shown in Brézis [**2**, Lemma 2], that

$$(8)\qquad (A'u_n,b_n)\geqq 0.$$

Let $v_n=n^{-1}u_n$. Dividing (7) by n^2 and observing (8) we get

$$\|n^{-1}b_n\|_H^2\leqq -t_n(A''v_n,n^{-1}b_n)+t_n n^{-1}(f,n^{-1}b_n).$$

This implies that the sequence $\{n^{-1}b_n\}$ is bounded in H. Thus also $A'v_n$ remains bounded in H as $n\to\infty$. By (4) and Rellich's compactness theorem we may pass to a subsequence and assure that $v_n\to v$ in X as well as a.e. on Ω. Hence $v\neq 0$.

On the other hand, by (1),

$$\int_\Omega \psi(u_n)u_n^2\,dx\leqq (b_n,u_n)\leqq -t_n(A''u_n-f,u_n).$$

It follows that

$$\int_\Omega \psi(u_n)v_n^2\,dx \leqq |(A''v_n - n^{-1}f, v_n)|;$$

consequently

$$\limsup \int_\Omega \psi(u_n(x))v_n^2(x)\,dx \leqq |(A''v, v)|.$$

But $\psi(u_n(x))=\psi(nv_n(x))\to+\infty$ where $v(x)\neq 0$. Hence $v(x)=0$ a.e., in contradiction to $v\neq 0$. We have proved that (5) holds for some $R>0$.

(ii) For $\lambda>0$ let $\overline{\beta_\lambda}$ be the Yosida approximation of $\bar\beta$ as described in §4. We assert the existence of a positive constant λ_0 having the property that

$$\begin{gathered} 0 \neq A'u + \overline{\beta_\lambda}(u) + t(A''u - f), \\ \forall u \in D(A') \quad \text{with } \|u\|_X = R, \qquad \forall 0 \leqq t \leqq 1, \quad \forall 0 < \lambda < \lambda_0. \end{gathered} \tag{9}$$

Suppose to each n there exist $u_n \in D(A')$, $\|u_n\|_X=R$, $0\leqq t_n\leqq 1$, and $0<\lambda_n<n^{-1}$, such that

$$0 = A'u_n + \overline{\beta_{\lambda_n}}(u_n) + t_n(A''u_n - f).$$

Then

$$0 = (A'u_n, \overline{\beta_{\lambda_n}}(u_n)) + \|\overline{\beta_{\lambda_n}}(u_n)\|_H^2 + t_n(A''u_n - f, \overline{\beta_{\lambda_n}}(u_n)).$$

Since $(A'u_n, \overline{\beta_{\lambda_n}}(u_n))\geqq 0$ and $t_n(A''u_n-f)$ remains bounded in H as $n\to\infty$, we infer as above the boundedness of the sequences $\{\overline{\beta_{\lambda_n}}(u_n)\}$ and $\{A'u_n\}$ in H. By (4) and the closedness of the mapping A' in H we may assume (after passage to subsequences) that

$$u_n \rightharpoonup u_0 \quad \text{in } X \text{ (and thus in } H), \tag{10}$$

$$A'u_n \rightharpoonup A'u_0 \quad \text{in } H, \tag{11}$$

and

$$\overline{\beta_{\lambda_n}}(u_n) \rightharpoonup z \quad \text{in } H. \tag{12}$$

Hence $u_0 \in D(A')$ and $\|u_0\|_X=R$. By (10), (12) and a result on Yosida approximations (e.g. Kato [**11**, Lemma 4.5]), $u_0 \in D(\bar\beta)$ and $z \in \bar\beta(u_0)$. Assuming further that $t_n\to t_0$ in $[0, 1]$, we infer that

$$\begin{gathered} 0 \in A'u_0 + \bar\beta(u_0) + t_0(A''u_0 - f), \\ u_0 \in D(A') \cap D(\bar\beta), \quad \|u_0\|_X = R, \end{gathered}$$

contradicting (5).

(iii) Let $A_0':X\to X^*$ be the bounded linear operator given by

$$(A_0'u, v) = \sum_{|\alpha|=|\beta|=1} \int_\Omega a_{\alpha\beta} D^\beta u\, D^\alpha v\,dx \qquad \forall u, v \text{ in } X.$$

It follows from (9) that

$$\text{(13)}\qquad \begin{gathered} 0 \neq A_0'u + \overline{\beta_\lambda}(u) + t(A''u - f), \\ \forall u \in X \quad \text{with } \|u\|_X = R, \qquad \forall 0 \leqq t \leqq 1, \quad \forall 0 < \lambda < \lambda_0. \end{gathered}$$

Indeed, assume

$$\text{(14)}\qquad 0 = A_0'u_1 + \overline{\beta_{\lambda_1}}(u_1) + t_1(A''u_1 - f)$$

for some $u_1 \in X$ with $\|u_1\|_X = R$, $0 \leqq t_1 \leqq 1$, and $0 < \lambda_1 < \lambda_0$. Then

$$0 = (A_0'u_1, \varphi) + (\overline{\beta_{\lambda_1}}(u_1), \varphi) + t_1(A''u_1 - f, \varphi)$$

$\forall \varphi \in C_0^\infty(\Omega)$. Since $\overline{\beta_{\lambda_1}}(u_1) + t_1(A''u_1 - f) \in H$ by the Lipschitz continuity of β_{λ_1}, it follows from regularity results on linear elliptic operators (e.g. Agmon [**1**, Theorem 9.8]), that $u_1 \in H^2(\Omega)$. But for $u_1 \in D(A')$, $A_0'u_1 = A'u_1$. Hence (14) constitutes a violation of (9).

As an immediate consequence of the Gårding inequality

$$(A_0'u, u) \geqq c_2 \|u\|_X^2, \qquad c_2 > 0, \quad \forall u \in X,$$

the Rellich theorem on compactness of the embedding $X \subset H$, and the monotonicity of $\overline{\beta_\lambda}$, the mapping

$$A_0' + \overline{\beta_\lambda} + t(A'' - f) : X \to X^*$$

is of type $(S)^+$ for each $0 \leqq t \leqq 1$ and $\lambda > 0$. By (13) and the Proposition, the equation $0 = A_0'u_\lambda + \overline{\beta_\lambda}(u_\lambda) + A''u_\lambda - f$ is solvable for $0 < \lambda < \lambda_0$, with $\|u_\lambda\|_X < R$. We conclude as above that $u_\lambda \in D(A')$; thus

$$\text{(15)}\qquad Au_\lambda + \overline{\beta_\lambda}(u_\lambda) = f.$$

(iv) In a similar way as in (ii) we now pass to the limit $\lambda \downarrow 0$ in (15) and obtain a desired solution of $Au + \bar{\beta}(u) \ni f$. Q.E.D.

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