ON A PROBLEM OF MAHLER IN THE GEOMETRY OF NUMBERS

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ABSTRACT. If $K$ is a convex body in $R_n$ and $K(t)$ is that part of $K$ which satisfies $|x_n| \leq t$, Mahler [2] has shown that $\Delta K(t)/t$ is a decreasing function of $t$, where $\Delta(K)$ is the critical determinant of $K$. We generalise Mahler's result in a way different from that conjectured by him.

1. Let $K$ be a closed, convex body in euclidean $n$-space $R_n$, symmetric in the origin 0. For a real and positive number $t$, denote by $K(t)$ that part of $K$ which satisfies $|x_n| \leq t$, where $x_n$ is the $n$th coordinate of a fixed coordinate system. If $V(K)$ denotes the volume of $K$, it was pointed out by Mahler [2] that $V(K(t))/t$ is a monotone decreasing function of $t$, and he conjectured the same holds true for the critical determinant of $K$. This number is defined as the infimum of the determinants of those lattices which do not contain an inner point of $K$, apart from 0. Mahler proved his conjecture for $n=2$. Bambah [1] showed that the same result holds true for the covering constant of $R_n$. This number is defined as the supremum of the determinants of those lattices $L$, for which the collection $X+K$, $X \in L$, covers $R_n$. The object of this note is to show these results have a natural extension to $n$ dimensions, along different lines to those conjectured by Mahler, the proof of which depends solely on the simple affine properties of the constants involved.

2. Let $m \leq n$ be a positive integer, and let $C$ denote any set in $R_m$. We assume that $R_m$ is embedded in $R_n$ in the usual manner. For a real and positive number $t$, we denote by $K(t)$ the set given by $K(t)=K \cap (tC \times R_{n-m})$, where $tC \times R_{n-m} = \{(x_1, \ldots, x_n) | (x_1, \ldots, x_m) \in tC\}$. We restrict the set $C$, throughout, to be such that $K(t)$ has Jordan content for all relevant $t$.

**Theorem 1.** $V(K(t))/t^m$ is a monotone decreasing function of $t$.

Let $F$ be a real-valued function of $K$ such that

(i) $F(T(K))=|\det T|F(K)$ for every nonsingular linear transformation $T$,
(ii) if $K_1 \subseteq K_2$, then $F(K_1) \leq F(K_2)$.

**Theorem 2.** If $m=n-1$, then $F(K(t))/t^{n-1}$ is a monotone decreasing function of $t$.

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Theorem 2 represents an extension of the results of Bambah and Mahler to higher dimensions, since the critical determinant and covering constant both satisfy (i) and (ii).

3. **Proof of Theorem 1.** Let \(0 < t < t', \) and let \(T\) denote the linear transformation given by \(x_i \rightarrow t'x_i/t\), for \(i = 1, \ldots, m\), so that \(\det T = (t'/t)^m\). Now

\[
V(K(t')) = \int_{R_m} V_1(x_1, \ldots, x_m) \, dx_1 \cdots dx_m,
\]

and

\[
V(T(K(t))) = \int_{R_m} V_2(x_1, \ldots, x_m) \, dx_1 \cdots dx_m
\]

where \(V_1(x_1, \ldots, x_m)\) is the \(n-m\) dimensional volume of the set \(S_1\),

\[
S_1 = \{(y_1, \ldots, y_n) \in K(t') \mid y_i = x_i \text{ for } i = 1, \ldots, m\};
\]

and \(V_2(x_1, \ldots, x_m)\) is the \(n-m\) dimensional volume of the set \(S_2\),

\[
S_2 = \{(y_1, \ldots, y_n) \in T(K(t)) \mid y_i = x_i \text{ for } i = 1, \ldots, m\}.
\]

Theorem 2 will follow if we can show \(V(S_1) \leq V(S_2)\), for all points \((x_1, \ldots, x_m) \in R_m\). The set \(T^{-1}(S_2)\) is a translation of \(S_2\), and so has volume \(V(S_2)\). Further, it is contained in \(K(t)\). Let \(V(x_1, \ldots, x_m)\) denote the \(n-m\) dimensional volume of the set

\[
\{(y_1, \ldots, y_n) \in X \mid y_i = x_i \text{ for } i = 1, \ldots, m\}.
\]

As \(K\) is symmetric in 0, \(V(x_1, \ldots, x_m) = V(-x_1, \ldots, -x_m)\). Therefore, by the Brunn-Minkowski theorem,

\[
V(0, \ldots, 0) \geq V(x_1, \ldots, x_m).
\]

There is clearly nothing to prove if \(S_1\) is the empty set, so we may assume that \(S_1\) is not empty, from which it follows

\[
S_1 = \{(y_1, \ldots, y_n) \in K \mid y_i = x_i \text{ for } i = 1, \ldots, m\},
\]

and

\[
T^{-1}(S_2) = \{(y_1, \ldots, y_n) \in K \mid y_i = tx_i/t' \text{ for } i = 1, \ldots, m\}.
\]

If \(S(0, \ldots, 0)\) denotes the section of \(K\) at the origin, namely

\[
S(0, \ldots, 0) = \{(y_1, \ldots, y_n) \in K \mid y_i = 0 \text{ for } i = 1, \ldots, m\},
\]

then, applying the Brunn-Minkowski theorem to the sets \(S(0, \ldots, 0), T^{-1}(S_2)\) and \(S_1\) and using the fact already established that \(V(0, \ldots, 0) \geq V(S_1)\), we obtain \(V(T^{-1}(S_2)) \geq V(S_1)\) and Theorem 1 follows.
4. Proof of Theorem 2. Let $P$ denote the tangent plane to $K$, at that point of the boundary of $K$ for which $x_1 = \cdots = x_{n-1} = 0$, $x_n = a > 0$, say. As the quantities in the theorem are unchanged by linear transformations of determinant 1 that leave the first $n-1$ coordinates fixed, we may assume, after application of such a transformation if necessary, that $P$ is given by $x_n = a$. Let $0 < t < t'$ and denote by $T$ the linear transformation given by

$$x_i \rightarrow t'x_i/t, \quad \text{for } i = 1, \cdots, n - 1.$$ 

If we can show that

(iii) $K(t') \subseteq T(K(t))$,

then by (i) and (ii),

$$F(K(t')) \leq F(T(K(t))) = (t'/t)^{n-1}F(K(t))$$

and the theorem is proved.

We assert that (iii) is true, for let $X \in K(t')$. We claim that the point $T^{-1}(X)$ is in $K$ and therefore also in $K(t)$. Otherwise, with $X = (x_1, \cdots, x_n)$,

$$T^{-1}(X) = (tx_1/t', \cdots, tx_{n-1}/t', x_n)$$

is not in $K$ and, since $0 < t/t' < 1$, so $(0, \cdots, 0, x_n)$ is not in $K$. However $P$ being a tangent plane to $K$ implies $|x_n| > a$, from which it follows that $X \notin K$, which is impossible. This contradiction proves Theorem 2.

References


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