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# ON THE GROWTH OF THE TAYLOR COEFFICIENTS OF AUTOMORPHIC FORMS

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ABSTRACT. The growth of the Taylor coefficients of an automorphic form of dimension $-2$ with respect to a Fuchsian group $\Gamma$ is related to the area integral $\iint_U |F|^s(1-|z|^2)^t\,dx\,dy$, and it is found that these coefficients must grow faster than a power of $n$. Moreover if $F\in H(p,\Gamma)$ then these coefficients must grow slower than a different power of $n$ and, in fact, $a_n/n$ is square summable if either $p=2$ or $1<p<\infty$ and $\Gamma$ is finitely generated of the second kind.

1. **Introduction.** Throughout $\Gamma$ shall stand for a Fuchsian group acting on the unit disk $U(=\{z:|z|<1\})$ of the complex plane. We shall assume that $\Gamma$ is of convergence type, i.e.,

$$\sum_{T\in\Gamma}|T'(z)|<\infty \quad \text{for all } z \text{ in } U. \tag{1.1}$$

We note that (1.1) is equivalent to the fact that the associated Riemann surface $U/\Gamma$ is hyperbolic (cf. Tsuji [**9**, p. 522]). If $F$ is a function defined on $U$, then we say that $F$ is an automorphic form of degree $-2$ if

$$F(Tz)T'(z)=F(z) \quad \text{for all } T \text{ in } \Gamma \text{ and } z \text{ in } U.$$

We define

$$\lambda^{-1}(z)=(1-|z|^2), \qquad d\omega(z)=\lambda^2(z)\,dx\,dy, \tag{1.2}$$

and note that $\lambda^{-1}(Tz)=|T'(z)|\lambda^{-1}(z)$ and $d\omega(Tz)=d\omega(z)$ for all conformal automorphisms $T$ of $U$ onto itself. If $\Omega$ is a fundamental region for $\Gamma$ whose boundary has two dimensional measure zero, then we define $H(p,\Gamma)$ $(1\leqq p<\infty)$ to be the space of holomorphic automorphic forms of

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degree $-2$ with respect to $\Gamma$ such that $\|F\|_p < \infty$ where

$$\|F\|_p^p = \iint_\Omega |F(z)|^p \lambda^{-p}(z)\, d\omega(z).$$

We note that $H(1, \Gamma) = \{0\}$ for a large class of groups $\Gamma$ (cf. [**5**, Theorem 3]). $H(\infty, \Gamma)$ is defined to be the space of holomorphic automorphic forms of degree $-2$ with respect to $\Gamma$ with the norm

$$\|F\|_\infty = \sup_{z \in U} |F(z)|\, \lambda^{-1}(z) < \infty.$$

In this paper we shall obtain estimates on the growth of the Maclaurin coefficients of holomorphic automorphic forms of dimension $-2$ by relating such growth to the finiteness of integrals of the form (cf. [**3**])

$$\iint_U |F|^s (1 - |z|^2)^t\, dx\, dy. \tag{1.3}$$

Although all of our work is done for the case $\Gamma$ of convergence type and automorphic forms of dimension $-2$, analogous results could be stated and proved for arbitrary $\Gamma$ and automorphic forms of dimension $-2q < -2$.

2. **Lower limits on the order of growth of the coefficients.** We shall first show that the Maclaurin coefficients of an automorphic form cannot grow "too" slowly.

THEOREM 1. *Suppose $\Gamma$ is such that $\sum_{T \in \Gamma} |T'(z)|^r = \infty$ for some $z$ in $U$. Let $F(z) = \sum_{n=0}^\infty a_n z^n$, $F \not\equiv 0$, be an arbitrary (holomorphic) automorphic form of degree $-2$ with respect to $\Gamma$, then, for any $t < r/2$, $a_n \neq O(n^t)$.*

*Note.* If $\sum_{T \in \Gamma} |T'(t)|^r = \infty$ for one $z$ in $U$ then $\sum_{T \in \Gamma} |T'(t)|^r = \infty$ for all $z$ in $U$.

PROOF. Suppose $a_n = O(n^t)$, then $\sum_{n=0}^\infty (|a_n|^2/(n+1)^{1+r}) < \infty$. Since $\Gamma(x)/\Gamma(x+a) \sim x^{-a}$, it follows that

$$\sum_{n=0}^\infty \frac{|a_n|^2\, \Gamma(r+1)\Gamma(n+1)}{\Gamma(n+r+2)} < \infty,$$

this in turn implies, since $\beta(p, q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$,

$$\sum_{n=0}^\infty |a_n|^2 \int_0^1 (1-u)^r u^n\, du < \infty.$$

Now upon letting $u = \rho^2$, where $|z| = \rho$, we see that Pareseval's formula implies that

$$\iint_U |F|^2(1-|z|^2)^r\,dx\,dy = \sum_{n=0}^{\infty} |a_n|^2 \int_0^1 (1-\rho^2)^r \rho^{2n+1}\,d\rho < \infty.$$

But this cannot happen if $F \not\equiv 0$ for

$$\infty > \iint_U |F|^2 (1-|z|^2)^r\,dx\,dy = \sum_{T\in\Gamma} \iint_{T\Omega} |F|^2 (1-|z|^2)^r\,dx\,dy$$
$$= \iint_\Omega |F|^2 (1-|z|^2)^r \sum_{T\in\Gamma} |T'(z)|^r\,dx\,dy,$$

since $|F|^2\,dx\,dy$ is $\Gamma$ invariant. Since $\sum_{T\in\Gamma} |T'(z)|^r \equiv \infty$ we have arrived at a contradiction and the proof of the theorem is complete.

Corollary 2. *Suppose $F$ is as above and $\Gamma$ is not cyclic hyperbolic, then* $a_n \neq 0(1)$.

Proof. If $\Gamma$ is not cyclic hyperbolic, then Beardon [**1**] proved that there exists an $r>0$ such that $\sum_{T\in\Gamma} |T'(t)|^r = \infty$; hence choosing $t=0$ in Theorem 1 completes the proof.

Remark. If $\Gamma$ is cyclic hyperbolic, Beardon [**1**, p. 475] showed that $\sum_{T\in\Gamma} |T'(t)|^r < \infty$ for all $r>0$, however in this case we have $a_n \neq O(n^{-\varepsilon})$ for all $\varepsilon>0$.

3. **Upper limits on the growth of the coefficients.** We note that in the above theorem and its corollary no assumption was made about the growth of $F$. If we make such an assumption, i.e., $F \in H(p, \Gamma)$, then it can be shown that the Maclaurin coefficients of $F$ cannot grow "too" fast. In order to prove this we first need

Lemma 3. *Let $F \in H(p, \Gamma)$* $(1 \leqq p < \infty)$. *Then* (1.3) *is finite with $s=p$ and $t=p-1$.*

Proof.

$$\iint_U |F|^p (1-|z|^2)^{p-1}\,dx\,dy = \iint_U |F\lambda^{-1}|^p (1-|z|^2)\,d\omega(z)$$
$$= \iint_\Omega |F\lambda^{-1}|^p \sum_{T\in\Gamma} (1-|Tz|^2)\,d\omega(z)$$
$$\leqq \left[\sup_U \sum_{T\in\Gamma} (1-|Tz|^2)\right] \|F\|_p^p.$$

This last term is finite by Theorem 3 of [7] and the fact $F \in H(p, \Gamma)$, if $\Gamma$ acts freely on $U$. If the origin is not a fixed point of $\Gamma$, then the proof of Theorem 3 of [7] goes through without modification to yield the desired result. If the origin is a fixed point of $\Gamma$, then, by conjugation, one can return to the previous case and again the result follows. This completes the proof of the lemma.

PROPOSITION 4. *Suppose* $F(z)=\sum_{n=0}^{\infty} a_n z^n \in H(p, \Gamma)$ $(1<p<\infty)$. *Then* (1.3) *is finite if* $s=1$ *and* $t>0$; *thus* $a_n=O(n^{1+t})$ *as* $n\to\infty$.

PROOF.

$$\iint_U |F|\,(1-|z|^2)^t\,dx\,dy = \iint_U |F|\,(1-|z|^2)^{(p-1)/p}(1-|z|^2)^{t-(p-1)/p}\,dx\,dy$$
$$\leqq \left(\iint_U |F|^p\,(1-|z|^2)^{p-1}\,dx\,dy\right)^{1/p}\cdot$$
$$\left(\iint_U (1-|z|^2)^{tp/(p-1)-1}\,dx\,dy\right)^{(p-1)/p} < \infty,$$

by Hölder's inequality, Lemma 3, and the fact $t>0$. The conclusion that $a_n=O(n^{1+t})$ as $n\to\infty$ now follows from Theorem 4 of [3].

REMARK. J. Lehner (private communication) has shown that if

$$F \in H(p, \Gamma) \quad (1 \leqq p < \infty), \qquad \text{then } a_n = O(n).$$

4. **Summability results.** From Proposition 4 we saw that $a_n=O(n^{1+t})$ for all $t>0$ if $\sum_{n=0}^{\infty} a_n z^n \in H(p, \Gamma)$. However even more is true if we assume $2\leqq p<\infty$ for arbitrary $\Gamma$ or $1<p<\infty$ and $\Gamma$ a finitely generated group. We note that if $\Gamma$ is finitely generated then $\Gamma$ is of convergence type if and only if $\Gamma$ is of the second kind. We first prove

LEMMA 5. *If* $\Gamma$ *is finitely generated of the second kind then, for* $1<p<\infty$,

$$\iint_\Omega \left(\sum_{T\in\Gamma} 1-|Tz|^2\right)^p d\omega(z) < \infty. \tag{4.1}$$

The idea of proof is that if one chooses $\Omega$ to be a Dirichlet region, then, $\partial\Omega\cap\partial U$ consists of free sides and parabolic cusps. On the free sides $\sum_{T\in\Gamma} |T'(z)|$ has a bounded supremum and so we must integrate $(1-|z|^2)^{p-2}\,dx\,dy$ which is finite. As for the parabolic cusps we need only recall that the hyperbolic area of a parabolic cusp is finite (cf. [4] for the details). Since $\|\cdot\|_p$ is independent of the choice made for the fundamental region the result follows.

THEOREM 6. *Suppose* $F(z)=\sum_{n=0}^{\infty} a_n z^n$. *Then* $\{a_n/n\} \in l_2$ *if either*
(i) $F \in H(2, \Gamma)$ *or*
(ii) $F \in H(p, \Gamma)$ $(1<p<\infty)$ *and* $\Gamma$ *is finitely generated.*

PROOF. In view of Parseval's formula for functions holomorphic in $U$ it suffices to show that (1.3) is finite with $s=2$ and $t=1$. Lemma 3 immediately yields the result in case (i). If $\Gamma$ is finitely generated, then, for $1<p\leqq q\leqq\infty$,

$$H(p, \Gamma) \subseteq H(q, \Gamma). \tag{4.2}$$

Assuming this for the moment, we see immediately that the theorem is true for $1<p\leqq 2$ in case (ii). If $p>2$, then

$$\begin{aligned}\iint_U |F|^2 (1-|z|^2)\,dx\,dy &= \iint_U |F\lambda^{-1}|^2 (1-|z|^2)\,d\omega(z)\\ &= \iint_\Omega |F\lambda^{-1}|^2 \sum_{T\in\Gamma} (1-|Tz|^2)\,d\omega(z)\\ &\leqq \left(\iint_\Omega |F\lambda^{-1}|^p\,d\omega(z)\right)^{2/p}\\ &\quad\times \left(\iint_\Omega \left(\sum_{T\in\Gamma}(1-|Tz|^2)\right)^{p/(p-2)} d\omega(z)\right)^{(p-2)/p}\\ &< \infty,\end{aligned}$$

by Lemma 5 and the fact $F \in H(p, \Gamma)$ with $p>2$. Thus the proof will be complete once (4.2) is established. This shall be done in §5.

In case $\Gamma$ is infinitely generated of convergence type and $F \in H(p, \Gamma)$ $(p\neq 2)$, similar methods to those above yield

PROPOSITION 7. *If* $\Gamma$ *is of convergence type and* $F(z)=\sum_{n=0}^{\infty} a_n z^n \in H(p, \Gamma)$, *then*
(i) $\{a_n n^{-r}\} \in l_2$ *for all* $r>1$ *if* $2<p<\infty$,
(ii) $\{a_n n^{-(p+2)/2p}\} \in l_2$ *if* $1<p\leqq 2$.

The idea of the proof is to show that (1.3) is finite if $s=2$ and $t>1$ in case (i) and $s=2$ and $t=2/p$ in case (ii). Then using the fact $\beta(p,q)=\Gamma(p)\Gamma(q)/\Gamma(p+q)$ and $\Gamma(x)/\Gamma(x+a)\sim x^{-a}$ the result follows by Parseval's formula.

5. **An auxiliary result.** In the proof of Theorem 6 we merely asserted (4.2). To prove this assertion we need to introduce the following notation analogous to that of [5]. Given $z, \zeta \in U$, we define

$$k(z, \zeta) = \pi^{-1}(1 - z\bar{\zeta})^{-2}$$

and

$$\alpha(z, \zeta) = \theta(k(z, \zeta), \Gamma)$$
$$= \sum_{T\in\Gamma} k(Tz, \zeta)T'(z).$$

It is known (cf. [7, Theorem 1]) that (every arrangement of) the series defining $\alpha(z, \zeta)$ converges, for fixed $\zeta$ in $U$, uniformly on compact subsets of $U$. Moreover since $k(z, \zeta)$ is a bounded analytic function for each $\zeta$ in $U$ it follows that $\alpha(z, \zeta) \in H(p, \Gamma)$ $(1<p\leqq\infty)$ whenever $\Gamma$ is finitely generated. Also we have

PROPOSITION 8. *Suppose* $\Gamma$ *is finitely generated of the second kind and* $F\in H(p, \Gamma)$ $(1<p<\infty)$. *Then*

$$\iint_{\Omega} F(z)\overline{\alpha(z, \zeta)}\, dx\, dy = F(\zeta). \tag{5.1}$$

PROOF. Formally we have

$$\iint_{\Omega} F(z)\overline{\alpha(z, \zeta)}\, dx\, dy = \iint_{\Omega} F(z) \sum_{T\in\Gamma} \overline{k(Tz, \zeta)}\; \overline{T'(z)}\, dx\, dy$$
$$= \iint_{U} F(z)\overline{k(z, \zeta)}\, dx\, dy = F(\zeta).$$

To complete the proof it suffices to show that

$$I = \iint_{U} |F(z)|\; |k(z, \zeta)|\, dx\, dy < \infty.$$

This follows immediately from the fact that $k(z, \zeta)$ is a bounded analytic function of $z$ for each $\zeta$ in $U$ and hence

$$I \leqq \|k(z, \zeta)\|_{\infty} \iint_{\Omega} |F(z)| \sum |T'(z)|\, dx\, dy$$
$$= \|k(z, \zeta)\|_{\infty} \iint_{\Omega} |F(z)\lambda^{-1}(z)| \left(\sum (1 - |Tz|^2)\right) d\omega(z).$$

Now Hölder's inequality, (4.1) and the fact that $F\in H(p, \Gamma)$ yields the desired result.

*Note* 1. If $\Gamma$ is finitely generated of the second kind then an analogous argument to that in [6] yields

$$0 \leqq \alpha(\zeta, \zeta) = \iint_{\Omega} |\alpha(z, \zeta)|^2 \, dx \, dy \leqq C(1 - |\zeta|^2)^{-2}, \tag{5.2}$$

$$|\alpha(z, \zeta)| \leqq C(1 - |z|^2)^{-1}(1 - |\zeta|^2)^{-1}. \tag{5.3}$$

*Note* 2. If one developed an $L_p$ theory for automorphic forms of dimension $-2q$, $q$ an arbitrary real ($q>1$), as in Drasin [2], then the above proposition has an analogous statement (viz. Theorem 3 of [2]).

We can now show

PROPOSITION 9. *If* $\Gamma$ *is finitely generated of the second kind then* $H(p, \Gamma) \subseteq (H(q, \Gamma))$ $(1<p\leqq q\leqq\infty)$.

PROOF. We first note that it suffices to show

$$H(p, \Gamma) \subseteq H(\infty, \Gamma), \qquad 1 < p < \infty, \tag{5.4}$$

for then, if $q>p$, it follows that

$$\begin{aligned} \iint_{\Omega} |F\lambda^{-1}|^q \, d\omega(z) &= \iint_{\Omega} |F\lambda^{-1}|^{q-p} \, |F\lambda^{-1}|^p \, d\omega(z) \\ &\leqq \|F\|_{\infty}^{q-p} \, \|F\|_p^p < \infty. \end{aligned}$$

To see that (5.4) holds we first assume that $1<p\leqq 2$. Then by (5.1) we have for $F \in H(p, \Gamma)$ and $1/p+1/p'=1$

$$\begin{aligned} |F(\zeta)| &= \left| \iint_{\Omega} F(z)\overline{\alpha(z, \zeta)} \, dx \, dy \right| \\ &\leqq \left( \iint_{\Omega} |F\lambda^{-1}|^p \, d\omega(z) \right)^{1/p} \left( \iint_{\Omega} |\alpha(z, \zeta)|^{p'} \, \lambda^{-p'} \, d\omega(z) \right)^{1/p'} \\ &= \|F\|_p \left( \iint_{\Omega} |\alpha(z, \zeta)|^{p'-2} \, |\alpha(z, \zeta)|^2 \, \lambda^{-p'+2} \, dx \, dy \right)^{1/p'} \\ &\leqq \|F\|_p \, C^{2-(1/p')}(1 - |\zeta|^2)^{-((p'-2)/p')}(1 - |\zeta|^2)^{2/p'} \\ &= \|F\|_p \, C^{2-(1/p')}(1 - |\zeta|^2)^{-1} \end{aligned}$$

by Hölder's inequality (5.2) and (5.3). This completes the proof in case $1<p\leqq 2$.

Now we note that an analogous argument based upon a similar reproducing formula for $p$-integrable holomorphic automorphic forms $G$ of dimension $-2q<-2$ (cf. Drasin [2]) would show that such forms are bounded if $1<p\leqq 2$, i.e.,

$$\sup_{z\in U} |G(z)|\,(1-|z|^2)^q < \infty.$$

Hence, we let $p>2$ and $F\in H(p,\Gamma)$, and choose an integer $m$ such that $m<p\leqq m+1$. It follows immediately that $F^m(z)$ is a $p/m$ integrable form of dimension $-2m$ and since $1<p/m\leqq 2$ the remark above yields that $|F^m(z)|(1-|z|^2)^m\leqq C$ for all $z$ in $U$, i.e.

$$\sup_{z\in U} |F(z)|\,(1-|z|^2) \leqq C^{1/m} < \infty.$$

This completes the proof.

REMARK 1. We note that it is easy to see that if one used an arbitrary system of factors of automorphy $\{\rho(z,T): T\in\Gamma\}$ instead of $T'(z)$ then the statements and results of all of the above theorems go through exactly as is.

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