

FREE DERIVATION MODULES AND A CRITERION FOR REGULARITY¹

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ABSTRACT. Let k be an algebraically closed field of characteristic zero, R an affine k -algebra. We prove that if the ideal of the variety of R can be generated by an S -sequence of forms in a polynomial ring S , and if the module of k -derivations of R into itself is a free R module, then R is regular.

1. Introduction. In this paper we investigate the following question: let R be the ring of polynomial functions on an affine variety V over an algebraically closed field k of characteristic zero, $D^*(R)$ the module of k -derivations of R into itself. Suppose $D^*(R)$ is a free R -module. Is R then a regular ring? This question was considered by Lipman [2] at the suggestion of Zariski. Lipman proved that R is normal, and gave examples which suggest the truth of an affirmative answer.

The object of this paper is to prove the

THEOREM. *Let R be a projective complete intersection, i.e., a residue class ring of a polynomial ring $S=k[X_1, X_2, \dots, X_n]$ over k (as above) by an ideal generated by an S -sequence of forms. Suppose $D^*(R)$ is free. Then R is regular (has finite global dimension).*

The proof makes use of the characterization of D^* as the module of row-relations on a certain matrix, namely the transpose of the Jacobian matrix associated with the polynomials generating the ideal of the variety V . We investigate the connection between the maximal-size minors of this matrix and the matrix obtained from a free basis for D^* by writing the elements of that basis as rows. We use these results in a certain Koszul complex related to D^* to complete the proof.

The main result of this paper generalizes an example due to Zariski [2, §7], that of a cone in three-space. The author wishes to thank Professor J. Eagon and Professor M. Hochster for many helpful conversations and insights.

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2. Notation and preliminaries. Let k be an algebraically closed field of characteristic zero, $S=k[X_1, X_2, \dots, X_n]$, I a radical ideal in S generated by an S -sequence of forms, $I=(F_1, F_2, \dots, F_r)$. Let $R=S/I$; we denote reduction modulo I by lower case letters. Let $J=(f_{ij})$ be the Jacobian matrix (with entries in R) associated to I ; thus

$$f_{ij} = \partial F_i / \partial X_j \text{ mod } I.$$

We identify d in $D^*(R)$ with the element $(dx_1, dx_2, \dots, dx_n)$ of R^n . Combining this identification with the chain rule, we note that $D^*(R)$ may be viewed as the module of row relations on the matrix J^t . If u_1, u_2, \dots, u_p generate $D^*(R)$, with $u_j=(u_{j1}, \dots, u_{jn}), 1 \leq j \leq p$, set $U=(u_{ij})$. We then have

PROPOSITION 1.1. $UJ^t=0$.

We assume from now on that $D^*(R)$ is free on generators u_1, u_2, \dots, u_p , and note [2, §2] that $p+r=n$.

There is an Euler-Poincaré mapping G which associates to each torsion module over R with finite free resolution a principal ideal of R (cf. [3]). Briefly, G is defined as follows. If M is a torsion module of homological dimension (dh) zero, $G(M)=(1)$. If $\text{dh}(M)=1, M=\text{coker } f: R^n \rightarrow R^m$, then $G(M)=(\det(f))$. If $\infty > \text{dh}(M) > 1$, resolve M by torsion modules T_j of homological dimension $\leq 1, 0 \rightarrow T_h \rightarrow T_{h-1} \rightarrow \dots \rightarrow T_2 \rightarrow T_1 \rightarrow M \rightarrow 0$, and set $G(M)=G(T_1)G(T_2)^{-1}G(T_3) \dots G(T_h)^{(-1)^{h+1}}$. By localizing at a rank one prime [3], one shows that G is well-defined, and is actually a principal ideal of R .

Let $\sigma=(\sigma_1, \sigma_2, \dots, \sigma_p), 1 \leq \sigma_1 < \sigma_2 < \sigma_3 < \dots < \sigma_p \leq n$, be an increasing sequence of p integers between 1 and n . Denote by Δ_σ the $p \times p$ minor of the $p \times n$ matrix U corresponding to columns $\sigma_1, \sigma_2, \dots, \sigma_p$, by τ_σ the $r \times r$ minor of J^t obtained by deleting rows $\sigma_1, \sigma_2, \dots, \sigma_p$.

PROPOSITION 1.2. *There is a unit $\mu \in R$ such that, for any $\sigma, \tau_\sigma = (-1)^{|\sigma|} \mu \Delta_\sigma$, where $|\sigma| = \sum_{i=1}^p \sigma_i$.*

PROOF. We show first that $(\tau_\sigma) = (\Delta_\sigma)$ by using the Euler-Poincaré (multiplicativity on exact sequences) property of the mapping G defined above. The result stems from an idea of Kramer [1].

Suppose for simplicity that $\sigma=(1, 2, \dots, p)$. There are two cases:

Case 1. $\tau_\sigma \neq 0$. Let ρ_i denote the i th row of the matrix J^t , and let P be the submodule of R^r generated by the ρ_i . Let M be the submodule of P generated by $\rho_{p+1}, \rho_{p+2}, \dots, \rho_n$. We obtain an exact sequence $0 \rightarrow K \rightarrow R^n \rightarrow P/M \rightarrow 0$ by factoring the canonical map $R^n \rightarrow P \rightarrow 0, e_i \rightarrow \rho_i$, where e_i are the usual basis elements of R^n . We claim $K = D^* \oplus e_{p+1} \oplus e_{p+2} \oplus \dots \oplus e_n$.

For

$$\begin{aligned}
 K &= \left\{ \sum_{i=1}^n a_i e_i \in R^n : \sum_{i=1}^n a_i \rho_i \in M \right\} \\
 &= \left\{ \sum_{i=1}^n a_i e_i : \text{there are } b_{p+1}, \dots, b_n \text{ in } R \right. \\
 &\quad \left. \text{with } \sum_{i=1}^p a_i \rho_i + \sum_{i=p+1}^n (a_i - b_i) \rho_i = 0 \right\} \\
 &= D^* \oplus e_{p+1} \oplus e_{p+2} \oplus \dots \oplus e_n.
 \end{aligned}$$

The proof of the last equality follows. Let $\sum_{i=1}^n a_i e_i \in K$. Then there are b_i in R such that

$$d = \sum_{i=1}^p a_i e_i + \sum_{i=p+1}^n (a_i - b_i) e_i \in D^*;$$

hence

$$\sum_{i=1}^n a_i e_i \in D^* \oplus e_{p+1} \oplus \dots \oplus e_n.$$

But if $\sum_{i=1}^n a_i e_i$ is an element of $D^* \oplus e_{p+1} \oplus e_{p+2} \oplus \dots \oplus e_n$, say

$$\sum_{i=1}^n a_i e_i = \sum_{i=1}^p r_i u_i + \sum_{i=p+1}^n b_i e_i,$$

then $\sum_{i=1}^p a_i e_i + \sum_{i=p+1}^n (a_i - b_i) e_i \in D^*$, completing the proof of the equality.

The sum is direct: suppose $\sum_{i=p+1}^n r_i e_i \in D^*$. Then $\sum_{i=p+1}^n r_i \rho_i = 0$, so $(r_{p+1}, r_{p+2}, \dots, r_n)$ is a solution of the $r \times r$ linear homogeneous system of equations with coefficient matrix whose columns are the ρ_i . The determinant of this matrix, τ_σ , is not zero; hence $r_{p+1} = r_{p+2} = \dots = r_n = 0$.

Thus K is free of rank n , and $G(P/M) = (\Delta_\sigma)$. Now from the exact sequence $0 \rightarrow P/M \rightarrow R^r/M \rightarrow R^r/P \rightarrow 0$, we get

$$(\tau_\sigma) = G(R^r/M) = G(R^r/P)G(P/M) = (\lambda)(\Delta_\sigma),$$

where λ is a unit in R . To see that λ is a unit in R , suppose it were not. λ is not a zero divisor, so suppose q is a grade one, rank one prime containing it. Since R is normal [2, §3], the singular locus of R has grade ≥ 2 . But the singular primes of R are also the primes containing the maximal-size minors of J , which are contained in $\text{Ann}(R^r/P)$. Hence $(R^r/P)_q = 0$, implying $(\lambda)_q = G(R^r/P)_q = G((R^r/P)_q) = (1)$, contradiction. This completes Case 1.

Case 2. $\tau_\sigma = 0$. Then there is a nontrivial solution to the $r \times r$ system of equations determined by the ρ_i as before, say $\sum_{i=p+1}^n r_i \rho_i = 0$. Then

$\sum_{i=p+1}^n r_i e_i \in D^*$, so there are t_i in R not all zero such that $\sum_{i=p+1}^n r_i e_i = \sum_{j=1}^p t_j u_j$. But this means that the t_j give a nontrivial solution to the $p \times p$ system of equations determined by the first p columns of U ; the determinant of this matrix is Δ_σ , hence $\Delta_\sigma = 0$, completing Case 2.

We have now shown that, for any σ , there is a unit μ_σ in R such that $\tau_\sigma = \mu_\sigma \Delta_\sigma$. To see that $\mu = \mu_\sigma$ is independent of σ , we use an argument from linear algebra. Consider the $n \times n$ matrix formed by writing the rows of J beneath the rows of U . Let K denote the quotient field of R ; by tensoring with K we may assume that the entries of this matrix are in K .

Thus we have an $n \times n$ matrix over K whose last r rows are orthogonal to the first p rows. We have the

LEMMA. *Let $A = (a_{ij})$ be an $n \times n$ matrix with entries in a field K . Denote the i th row of A by A_i , and assume that $A_i A_j^t = 0$ for $1 \leq i \leq s$, $s+1 \leq j \leq n$. Further, assume that A_1, A_2, \dots, A_s and $A_{s+1}, A_{s+2}, \dots, A_n$ are linearly independent over K . Then there is an element μ of K such that $\tau_\sigma = (-1)^{|\sigma|} \mu \Delta_\sigma$ for any choice of s indices as before, where Δ_σ is the $s \times s$ minor corresponding to σ from the first s rows of A , and τ_σ the corresponding $r \times r$ minor from the last $r = n - s$ rows of A .*

To prove the lemma, assume first that

$$A = \begin{pmatrix} I_s & M \\ N & I_r \end{pmatrix},$$

where I_s (resp. I_r) is the $s \times s$ (resp. $r \times r$) identity matrix, so that $N = -M^t$. Fix σ and assume $\sigma_k \leq s < \sigma_{k+1}$. Then

$$\Delta_\sigma = (-1)^{\sum_{i=1}^k \sigma_i - (k^2 - 3k)/2} \Delta_\sigma^*,$$

where Δ_σ^* is an $(s - k) \times (s - k)$ minor of M .

If ξ denotes the set of r indices complementary to σ , we have $\tau_\sigma = (-1)^m \Delta_\sigma^*$, where m is given by the formula

$$m = (s - k)(r - s + k) + \sum_{i=1}^{r-s+k} \xi_{s-k+i} - \frac{s(r - s + k)^2 - 3(r - s + k)}{2} + (s - k),$$

using the fact that $N = -M^t$.

Hence $\tau_\sigma = \pm (\text{sgn } \sigma) \Delta_\sigma$, where the first sign is independent of σ , as follows by checking each term of m and observing that those terms which depend upon σ do so exactly as $\text{sgn } \sigma = (-1)^{|\sigma|}$.

If A is now any $n \times n$ matrix satisfying the hypotheses of the lemma,

assume (as we may) that for $\sigma(0)=(1, 2, \dots, s)$, $\Delta_{\sigma(0)} \neq 0$, $\tau_{\sigma(0)} \neq 0$. Write

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} is the $s \times s$ matrix whose determinant is $\Delta_{\sigma(0)}$, A_{22} the $r \times r$ matrix whose determinant is $\tau_{\sigma(0)}$. Then the matrix

$$A' = \begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} I_s & A_{11}^{-1}A_{12} \\ A_{22}^{-1}A_{21} & I_r \end{pmatrix}$$

satisfies the hypotheses of the lemma, hence its conclusion. It is immediate that the $s \times s$ minors satisfy $\Delta'_\sigma = (\Delta_{\sigma(0)})^{-1} \Delta_\sigma$, and that the $r \times r$ minors satisfy $\tau'_\sigma = (\tau_{\sigma(0)})^{-1} \tau_\sigma$.

Combining these results, we obtain

$$\begin{aligned} \tau_\sigma &= \tau_{\sigma(0)} \tau'_\sigma = \tau_{\sigma(0)} [\pm(\text{sgn } \sigma) \Delta'_\sigma] \\ &= \tau_{\sigma(0)} [\pm(\text{sgn } \sigma) (\Delta_{\sigma(0)})^{-1}] = (\tau_{\sigma(0)} / \Delta_{\sigma(0)}) (\pm \text{sgn } \sigma) \Delta_\sigma. \end{aligned}$$

This completes the proof of the lemma.

To finish the proof of the proposition, we need only remark that our previous argument shows that $\tau_{\sigma(0)} / \Delta_{\sigma(0)}$ is a unit in R , not merely in the quotient field of R . Q.E.D.

3. Main result. Before stating the theorem and completing its proof, we remark that the property of homogeneity implies (Euler's formula) that $x=(x_1, x_2, \dots, x_n)$ is an element of $D^*(R)$, where x_i =image of X_i in $R=S/I$.

THEOREM. *Let $R=S/I$ be a reduced projective complete intersection. Assume that $D^*(R)$ is free. Then R is regular (has finite global dimension). (Note: it follows that I must be generated by forms of degree one.)*

PROOF. Suppose R is not regular. We may then assume that x is a member of a free basis for $D^*(R)$ (cf. Seidenberg [4]). We apply Proposition 1.2 to obtain a contradiction.

Let $\wedge S^n$ be the exterior algebra on S^n , where $S^n=SV_1+SV_2+\dots+SV_n$ with V_i the usual basis for S^n . Define maps $d_i: \wedge S^n \rightarrow \wedge S^n$ by

$$d_i V_j = F_{ij} / m_i \quad (m_i = \text{deg } F_i)$$

and require that each d_i be a homogeneous derivation of degree (-1) on S^n (i.e., $d_i(p \wedge q) = d_i(p) \wedge q + (-1)^{\text{deg } p} p \wedge d_i(q)$). Finally, let $V = V_1 \wedge V_2 \wedge \dots \wedge V_n$ generate $\wedge^n S^n$, and set $D_k = d_1 \circ d_2 \circ \dots \circ d_k$, $1 \leq k \leq n$. We shall show that the assumption on x implies that $V \in M(\wedge^n S^n)$, where M is

the ideal generated by X_1, X_2, \dots, X_n in S , contradicting the definition of V .

Fix a free basis x, u_2, u_3, \dots, u_p of $D^*(R)$. Let X denote the element (X_1, X_2, \dots, X_n) of S^n . The quotient map $S \rightarrow R$ induces a homomorphism $S^n \rightarrow R^n$ which maps X on x . For $2 \leq i \leq p$, let U_i be an element of S^n which is mapped on u_i by this induced homomorphism.

We also have an induced homomorphism of exterior algebras; let $[D_r V]$ denote the image under this homomorphism of $D_r V$ in $\wedge^p R^n$. Then the conclusion of Proposition 1.2 can be stated

$$x \wedge u_2 \wedge u_3 \wedge \dots \wedge u_p = [D_r V].$$

This equality in $\wedge R^n$ can be rewritten in $\wedge S^n$ as $X \wedge U_2 \wedge U_3 \wedge \dots \wedge U_p = D_r V - \sum_{i=1}^r F_i E_i^p$, where $E_i^p \in \wedge^p S^n, 1 \leq i \leq r$.

Since $X \wedge V = 0$, we have

$$\begin{aligned} 0 &= D_r(X \wedge V) = d_1 \circ d_2 \circ \dots \circ d_r(X \wedge V) \\ &= d_1 \circ d_2 \circ \dots \circ d_{r-1}(d_r(X) \wedge V \pm X \wedge d_r(V)) \\ &= d_1 \circ d_2 \circ \dots \circ d_{r-1}(F_r V \pm X \wedge d_r(V)) \\ &= \dots = F_r D_{r-1}(V) \\ &\quad \pm \text{terms in } (F_1, F_2, \dots, F_{r-1}) \wedge S^n \pm X \wedge D_r(V). \end{aligned}$$

But

$$\begin{aligned} X \wedge D_r(V) &= X \wedge (X \wedge U_2 \wedge \dots \wedge U_p) - \sum_{i=1}^r F_i (X \wedge E_i^p) \\ &= - \sum_{i=1}^r F_i (X \wedge E_i^p). \end{aligned}$$

Combining yields $F_r(D_{r-1}(V) \pm X \wedge E_r^p) \in (F_1, F_2, \dots, F_{r-1}) \wedge S^n$. But F_1, F_2, \dots, F_r is an S -sequence on $\wedge S^n$, hence $D_{r-1}(V) \pm (X \wedge E_r^p) = \sum_{i=1}^{r-1} F_i E_i^{p+1}$, where $E_i^{p+1} \in \wedge^{p+1} S^n, 1 \leq i \leq r-1$.

Repeat the argument:

$$\begin{aligned} 0 &= D_{r-1}(X \wedge V) = D_{r-2}(F_{r-1} V \pm X \wedge D_{r-1}(V)) \\ &= \dots = F_{r-1} D_{r-2}(V) + \text{terms in } (F_1, \dots, F_{r-2}) \wedge S^n \pm X \wedge D_{r-1}(V) \\ &= F_{r-1}(D_{r-2}(V) \pm X \wedge E_{r-1}^{p+1}) + \text{terms in } (F_1, \dots, F_{r-2}) \wedge S^n. \end{aligned}$$

Hence $D_{r-2}(V) \pm X \wedge E_{r-1}^{p+1} = \sum_{i=1}^{r-2} F_i E_i^{p+2}$.

Repeating the argument $r-1$ times, we obtain $D_1(V) \pm X \wedge E_2^{n-2} = F_1 E_1^{n-1}$. Then

$$\begin{aligned} 0 &= D_1(X \wedge V) = d_1(X \wedge V) \\ &= F_1 V \pm X \wedge d_1(V) = F_1(V - X \wedge E_1^{n-1}). \end{aligned}$$

Hence $V = X \wedge E_1^{n-1} \in M(\wedge^n S^n)$, contradicting the choice of V . Q.E.D.

REFERENCES

1. H. Kramer, *Eine Bemerkung zu einer Vermutung von Lipman*, Ark. Mat. **20** (1969), 30–35.
2. J. Lipman, *Free derivation modules on algebraic varieties*, Amer. J. Math. **87** (1965), 874–898. MR **32** #4130.
3. R. E. MacRae, *On an application of the Fitting invariants*, J. Algebra **2** (1965), 153–169. MR **31** #2296.
4. A. Seidenberg, *Differential ideals in rings of finitely generated type*, Amer. J. Math. **89** (1967), 22–42. MR **35** #2902.

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