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PRIME GENERALIZED ALTERNATIVE RINGS I WITH NONTRIVIAL IDEMPOTENT

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ABSTRACT. A generalized alternative ring I is a nonassociative ring R in which the identities $(wx, y, z)+(w, x, [y, z])-w(x, y, z)-(w, y, z)x$; $([w, x], y, z)+(w, x, yz)-y(w, x, z)-(w, x, y)z$; and (x, x, x) are identically zero. It is demonstrated here that if R is a ring of this type with characteristic different from two and three, then R semiprime with idempotent e implies that R has a Peirce decomposition relative to e. Furthermore, if R is prime and $e\neq 0, 1$; then R must be alternative.

1. **Introduction.** Let R be a nonassociative ring. As is customary, for $x, y, z \in R$ we denote by (x, y, z) the associator $(x, y, z)=(xy)z-x(yz)$ and by $[x, y]$ the commutator $[x, y]=xy-yx$. R is called power-associative if for every $x \in R$ the subring generated by x is associative.

In [2] Kleinfeld defines a generalized alternative ring I to be a nonassociative ring R such that for all $w, x, y, z \in R$ the following identities are satisfied:

(1) $(wx, y, z)+(w, x, [y, z])=w(x, y, z)+(w, y, z)x$,

(2) $([w, x], y, z)+(w, x, yz)=y(w, x, z)+(w, x, y)z$,

(3) $(x, x, x)=0$.

It is easily verified that a ring of this type is power-associative.

Consider R to be a power-associative ring of characteristic different from two and define $x \circ y=\frac{1}{2}(xy+yx)$ for $x, y \in R$. If R contains an idempotent e, then Albert has shown in [1] that $R=R_1(e)+R_{1/2}(e)+R_0(e)$ where $R_i(e)=\{x \in R : x \circ e=ix\}$. In fact, $ex=x=xe$ for $x \in R_1(e)$ and $ex=0=xe$ for $x \in R_0(e)$. This decomposition of R is known as the Albert decomposition. Throughout this work we will denote $R_i(e)$ by just R_i.

Suppose now one also has $(R, e, e)=(e, R, e)=(e, e, R)=0$. If, as in the associative case, one takes $x=exe+(ex-exe)+(xe-exe)+(x-ex-xe+exe)$, one sees that $R=R_{11}+R_{10}+R_{01}+R_{00}$ where $R_{ij}=\{x \in R : ex=ix, xe=jx\}$. This further decomposition of R is referred to as the Peirce decomposition.

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A nonassociative ring R is said to be semiprime if R contains no nonzero ideal I such that $I^2=0$. If, in addition, R contains no nonzero ideals I and H such that $IH=0$, then R is said to be prime.

2. **Preliminaries.** Henceforth we assume R to be a generalized alternative ring I with characteristic different from two and three. Under the additional assumption that R contains an idempotent e, we will make use of the following results established by Kleinfeld in **[2]**:

(i) $(e, R, e)=0$.

(ii) In the Albert decomposition, $R_{1/2}R_i \subseteq R_{1/2}$ and $R_iR_{1/2} \subseteq R_{1/2}$ for $i=0, 1$. Furthermore, $B=\{x \in R_{1/2}: xR \subseteq R_{1/2}, Rx \subseteq R_{1/2}\}$ is an ideal of R such that $(e, e, R) \subseteq B$ and every element of B squares to zero.

(iii) If R permits a Peirce decomposition, then for $i, j, k, l=0$ or 1, $R_{ij}R_{kl}=0$ when $j \neq k$ and $R_{ij}R_{jl} \subseteq R_{il}$, with two exceptions, namely $R_{01}R_{01} \subseteq R_{10}$ and $R_{10}R_{10} \subseteq R_{01}$. Furthermore, if $(R_{01}, R_{11}, R_{11})= (R_{11}, R_{11}, R_{10})=(R_{10}, R_{00}, R_{00})=(R_{00}, R_{00}, R_{01})=0$, then $I_i= \sum (R_{ii}, R_{ii}, R_{ii})+\sum (R_{ii}, R_{ii}, R_{ii})R_{ii}$ for $i=0, 1$ are ideals of R.

The following identities will also be necessary for our calculations:

(4) $(wx, y, z)-(w, xy, z)+(w, x, yz)=w(x, y, z)+(w, x, y)z$,

(5) $(w, xy, z)-(w, x, zy)+(w, x, y)z-(w, y, z)x=0$,

(6) $(w, xy, z)-(xw, y, z)+w(x, y, z)-y(w, x, z)=0$,

(7) $(x, x, yz)=y(x, x, z)+(x, x, y)z$.

A straightforward verification shows that (4), known as the Teichmüller identity, holds for all w, x, y, z in any nonassociative ring. Identity (5) is obtained by subtracting (4) from (1), while (6) is obtained by subtracting (4) from (2). Identity (7) follows from letting $w=x$ in identity (2).

3. **Semiprime and prime rings.**

THEOREM 1. *Let R be a generalized alternative ring I with characteristic different from two and three. If R contains an idempotent e, then $I= (e, e, R)$ is an ideal of R such that $I^2=0$. Furthermore, $[I, R]=0$ and $R_{1/2}I=(R_{1/2})^2I=0$.*

PROOF. By (ii), $B=\{x \in R_{1/2}: xR \subseteq R_{1/2}, Rx \subseteq R_{1/2}\}$ is an ideal of R such that $(e, e, R) \subseteq B$ and every element of B squares to zero. Let $K= (e, e, B) \subseteq B$. We first show that K is an ideal of R such that $0=K^2= [K, R]=R_{1/2}K=(R_{1/2})^2K$.

We begin by making the following observations. Let $(e, y, z)= a_1+a_{1/2}+a_0$ and $(e, e, [y, z])=b_{1/2}$ where $a_i \in R_i$ for $i=0, \frac{1}{2}, 1$ and $b_{1/2} \in R_{1/2}$. Then (1) yields $(e, y, z)+(e, e, [y, z])=e(e, y, z)+(e, y, z)e$ or $a_1+a_{1/2}+a_0+b_{1/2}=a_1+ea_{1/2}+a_1+a_{1/2}e$, whence $a_1=a_0=b_{1/2}=0$. Since from [1] we know $y, z \in R_{1/2}$ implies $y \circ z \in R_1+R_0$, we also have $0=(e, e, yz+zy)+(e, e, yz-zy)=2(e, e, yz)$ or $(e, e, yz)=0$ if $y, z \in R_{1/2}$.

Now if we let $w=x=z$ in (2) and then apply (3), we have $(z, z, yz)=(z, z, y)z$. If we then let $w=x=y$ in (5) and apply (3), we have $(y, y^2, z)=2(y, y, z)y$. Taking $y=e$ and $z=x$, this gives $(e, e, x)=2(e, e, x)e$. Let $b=(e, e, x)$. Then $b=2be$ and, since $b=eb+be$, this implies $eb=\frac{1}{2}b=be$. In particular, we have $ek=\frac{1}{2}k=ke$ for every $k \in K$. Now taking $k, k' \in K \subseteq R_{1/2}$ and using our last observation above together with (7), we have $0=(e, e, kk')=k(e, e, k')+(e, e, k)k'=\frac{1}{4}kk'+\frac{1}{4}kk'=\frac{1}{2}kk'$ or $K^2=0$.

We next show K to be an ideal of R. That K is additive is clear. Let $b \in B$, $k \in K$. Since $B \subseteq R_{1/2}$ implies $(e, e, B^2)=0$, using (7) and the fact that $K^2=0$ we have $0=(e, e, bk)=b(e, e, k)+(e, e, b)k=b(e, e, k)=\frac{1}{4}bk$ or $BK=0$. Since $b, b' \in B$ implies $(b+b')^2=0$ or $bb'=-b'b$, it follows too that $KB=0$. Let $x_i \in R_i$ for $i=0, \frac{1}{2}, 1$. Noting that previous calculations show if $b'=(e, e, x_{1/2})$, then $(e, e, 4b')=b'=(e, e, x_{1/2})$, we now use (7), the fact that $BK=0=KB$, and the fact that B is an ideal of R to compute as follows:

$$(e, e, b)x_i = (e, e, bx_i) - b(e, e, x_i) = (e, e, bx_i) \in K \quad \text{for } i = 0, 1;$$

$$\begin{aligned}(e, e, b)x_{1/2} &= (e, e, bx_{1/2}) - b(e, e, x_{1/2}) = (e, e, bx_{1/2}) - b(e, e, 4b') \\ &= (e, e, bx_{1/2}) \in K;\end{aligned}$$

$$x_i(e, e, b) = (e, e, x_ib) - (e, e, x_i)b = (e, e, x_ib) \in K \quad \text{for } i = 1, 1;$$

$$\begin{aligned}x_{1/2}(e, e, b) &= (e, e, x_{1/2}b) - (e, e, x_{1/2})b = (e, e, x_{1/2}b) - (e, e, 4b')b \\ &= (e, e, x_{1/2}b) \in K.\end{aligned}$$

Thus it follows K is an ideal of R.

Suppose now we are given $x \in R$. Let $x=x_1+x_{1/2}+x_0$ where $x_i \in R_i$ for $i=0, \frac{1}{2}, 1$. Then, using (7) while keeping in mind that $k=(e, e, 4k)$ for $k \in K$ and $0=(e, e, [y, z])$ for $y, z \in R$, one has for $i=0, 1$ that $x_ik=x_i(e, e, 4k)+(e, e, x_i)(4k)=4(e, e, x_ik)=(4k)(e, e, x_i)+(e, e, 4k)x_i=kx_i$. Also keeping in mind that $0=(e, e, yz)$ for $y, z \in R_{1/2}$ and $BK=0=KB$, one has $0=(e, e, x_{1/2}k)=x_{1/2}(e, e, k)+(e, e, x_{1/2})k=x_{1/2}(e, e, k)=\frac{1}{4}x_{1/2}k$ as well as $0=(e, e, kx_{1/2})=k(e, e, x_{1/2})+(e, e, k)x_{1/2}=\frac{1}{4}kx_{1/2}$. Thus $[K, R]=0$ and, in particular, $R_{1/2}K=0$.

Next let $x, y \in R_{1/2}$ and $k \in K$. Then (1) gives $(xy, e, k)+(x, y, [e, k])=x(y, e, k)+(x, e, k)y$. But $[K, R]=0$ implies $(x, y, [e, k])=0$, while K an ideal of R with $R_{1/2}K=0$ implies $x(y, e, k)=0=(x, e, k)y$. Hence $(xy, e, k)=0$. Let $xy=a_1+a_{1/2}+a_0$ where $a_i \in R_i$ for $i=0, \frac{1}{2}, 1$. Then $0=(xy, e, k)=[(xy)e]k-\frac{1}{2}(xy)k=(a_1+a_{1/2}e)k-\frac{1}{2}a_1k-\frac{1}{2}a_0k=a_1k-\frac{1}{2}a_1k-\frac{1}{2}a_0k$, using the fact from (ii) that $R_{1/2}R_1 \subseteq R_{1/2}$. Thus $a_1k=a_0k$.

Now from our initial observations we have $(e, x, y) \in R_{1/2}$. Hence

$(e, x, y)=(ex)y-e(xy)=(ex)y-a_1-ea_{1/2}$ gives $[(ex)y]_1=a_1$ and $[(ex)y]_0=0$. Then $a_1+a_{1/2}+a_0=xy=(ex)y+(xe)y$ implies $[(xe)y]_1=0$. Thus $(ex)y \in R_1+R_{1/2}$ while $(xe)y \in R_{1/2}+R_0$. Now since from (ii) we know $xe \in R_{1/2}$, our previous argument shows $[(xe)y]_1k=[(xe)y]_0k$. But $[(xe)y]_1=0$, so $[(xe)y]_1k=0=[(xe)y]_0k$. Hence $[(xe)y]k=[(xe)y]_1k+[(xe)y]_{1/2}k+[(xe)y]_0k=0$, since $R_{1/2}K=0$. In similar fashion we have $[(ex)y]k=0$. But then $x \in R_{1/2}$ gives $(xy)k=[(xe+ex)y]k=[(xe)y]k+[(ex)y]k=0$ or $(R_{1/2})^2K=0$.

Finally, it is clear that $K=(e, e, B)\subseteq(e, e, R)$. But we have also shown above that if $b=(e, e, x_{1/2})$, then $(e, e, 4b)=b=(e, e, x_{1/2})$. Thus $(e, e, R)\subseteq(e, e, 4B)=K$, that is $K=(e, e, R)$.

COROLLARY. *Let R be a generalized alternative ring I with characteristic different from two and three. If R contains an idempotent e, then R semiprime implies that R has a Peirce decomposition relative to e.*

PROOF. Linearization of (3) gives $(y, x, x)+(x, y, x)+(x, x, y)=0$. By (i), one always has $(e, R, e)=0$. Since, by Theorem 1, $I=(e, e, R)$ is an ideal of R such that $I^2=0$, R semiprime now implies in addition that $(e, e, R)=0=(R, e, e)$.

THEOREM 2. *Let R be a prime generalized alternative ring I with characteristic different from two and three. If R contains an idempotent $e\neq 0, 1$; then R is alternative.*

PROOF. Since R a prime ring implies that R is semiprime, by the above Corollary R has a Peirce decomposition relative to e. Throughout this proof we use the convention that $w_{ij}, x_{ij}, y_{ij}, z_{ij} \in R_{ij}$ for $i,j=0$ or 1. Then using the multiplication table described by (iii), (5) gives $(x_{11}, ey_{11}, z_{10})-(x_{11}, e, z_{10}y_{11})+(x_{11}, e, y_{11})z_{10}-(x_{11}, y_{11}, z_{10})e=0$ and $(x_{00}, y_{00}e, z_{01})-(x_{00}, y_{00}, z_{01}e)+(x_{00}, y_{00}, e)z_{01}-(x_{00}, e, z_{01})y_{00}=0$, that is $(R_{11}, R_{11}, R_{10})=0=(R_{00}, R_{00}, R_{01})$. Similarly (6) and (iii) yield $(x_{01}, y_{11}e, z_{11})-(y_{11}x_{01}, e, z_{11})+x_{01}(y_{11}, e, z_{11})-e(x_{01}, y_{11}, z_{11})=0$ and $(x_{10}, ey_{00}, z_{00})-(ex_{10}, y_{00}, z_{00})+x_{10}(e, y_{00}, z_{00})-y_{00}(x_{10}, e, z_{00})=0$, that is $(R_{01}, R_{11}, R_{11})=0=(R_{10}, R_{00}, R_{00})$. Thus by (iii), $I_i=\sum (R_{ii}, R_{ii}, R_{ii})+\sum (R_{ii}, R_{ii}, R_{ii})R_{ii}$ are ideals for $i=0, 1$.

We now use a technique which was first used by Kleinfeld for prime associator-dependent rings. Consider $H=R_{10}R_{01}+R_{10}+R_{01}+R_{01}R_{10}$. Using the multiplication table described by (iii), it follows that to show H is an ideal of R it suffices to verify that $R_{10}R_{01}$ is an ideal of R_{11} and $R_{01}R_{10}$ is an ideal of R_{00}. But from (5) and (iii) we have $(x_{11}, y_{10}e, z_{01})-(x_{11}, y_{10}, z_{01}e)+(x_{11}, y_{10}, e)z_{01}-(x_{11}, e, z_{01})y_{10}=0$ or $(R_{11}, R_{10}, R_{01})=0$, while (6) and (iii) yield $(x_{10}, ey_{01}, z_{11})-(ex_{10}, y_{01}, z_{11})+x_{10}(e, y_{01}, z_{11})-y_{01}(x_{10}, e, z_{11})=0$ or $(R_{10}, R_{01}, R_{11})=0$. Thus $R_{10}R_{01}$ is an ideal of R_{11}.

Analogously one has $(R_{00}, R_{01}, R_{10})=0=(R_{01}, R_{10}, R_{00})$ or $R_{01}R_{10}$ is an ideal of R_{00}.

Now (5) and (iii) give $(w_{11}, x_{10}y_{11}, z_{11})-(w_{11}, x_{10}, z_{11}y_{11})+(w_{11}, x_{10}, y_{11})z_{11}-(w_{11}, y_{11}, z_{11})x_{10}=0$ or $(R_{11}, R_{11}, R_{11})R_{10}=0$. Also, $[(R_{11}, R_{11}, R_{11})R_{11}]R_{10}\subseteq((R_{11}, R_{11}, R_{11}), R_{11}, R_{10})+(R_{11}, R_{11}, R_{11})(R_{11}R_{10})=0$, using (iii) and earlier calculation that $(R_{11}, R_{11}, R_{10})=0$. Hence $I_1R_{10}=0$. Then $I_1(R_{10}R_{01})\subseteq(I_1, R_{10}, R_{01})+(I_1R_{10})R_{01}=0$, since we have shown previously $(R_{11}, R_{10}, R_{01})=0$. From these last calculations and (iii), it now follows that $I_1H=0$. Similarly one may show $I_0H=0$.

Suppose $H=0$. Then $R_{10}=0=R_{01}$ implies R_{11} and R_{00} are ideals of R such that $R_{11}R_{00}=0$. Since R is a prime ring and $e\in R_{11}$, we must have $R_{00}=0$. But then $R=R_{11}$ implies $e=1$, contrary to assumption. Thus it must be the case $H\neq 0$. But $H\neq 0$ and R a prime ring imply $I_1=0=I_0$, or that R_{11} and R_{00} are associative subrings of R.

We may now join the calculations on p. 316 of [2] to conclude that for any $x_1, x_2, x_3\in R$ one has $(x_1, x_2, x_3)=\operatorname{sgn}\sigma(x_{\sigma 1}, x_{\sigma 2}, x_{\sigma 3})$ where σ is any permutation of three elements. Since this implies R is alternative, the proof is complete.

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