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A REPRESENTATION FORMULA FOR HARMONIC FUNCTIONS

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ABSTRACT. We give a formula to reconstruct certain entire harmonic functions from their values on some lattice points.

1. **Introduction and results.** In [1], Boas proved the following uniqueness theorem for harmonic functions.

THEOREM A. *Let $u(z)$ be a real-valued entire harmonic function of exponential type less than π such that $u(m)=0$ and $u(m+i)=0$ for $m=0, \pm 1, \pm 2, \cdots$. Then $u(z)\equiv 0$.*

He also asked if it is possible to reconstruct an entire harmonic function of exponential type less than π from its values on the lattice points m, $m+i$. We have

THEOREM 1. *Let $u(z)$ be a real-valued entire harmonic function of exponential type $\tau \leqq \pi$ such that $u(x)$ and $u(x+i)$ are in $L^2(-\infty, \infty)$. Then*

$$
\begin{aligned}
u(z) &= u(x+iy) \\
&= \sum_{n=-\infty}^{\infty} \frac{u(n)}{2\pi} \int_{-\pi}^{\pi} \frac{\sinh t(1-y)}{\sinh t} e^{it(x-n)}\, dt \\
&\quad + \sum_{n=-\infty}^{\infty} \frac{u(n+i)}{2\pi} \int_{-\pi}^{\pi} \frac{\sinh ty}{\sinh t} e^{it(x-n)}\, dt \\
&\quad + c_1 e^{\pi x} \sin \pi y + c_2 e^{-\pi x} \sin \pi y
\end{aligned}
\tag{1}
$$

where the series converges uniformly in every strip $|y| \leqq K < \infty$. Furthermore, if $\tau < \pi$ then $c_1 = c_2 = 0$, and if $\tau = \pi$ then

$$c_1 = \lim_{x\to\infty} \frac{e^{-\pi x} u(x+iy)}{\sin \pi y} \quad \text{and} \quad c_2 = \lim_{x\to\infty} \frac{e^{-\pi x} u(-x+iy)}{\sin \pi y}$$

for any y, $0<y<1$.

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Hence, we have the following

COROLLARY. *Let $u(z)$ be a real-valued entire harmonic function of exponential type equal to π such that $u(x)$ and $u(x+i)$ are in $L^2(-\infty, \infty)$, $u(m)=0$ and $u(m+i)=0$ for $m=0, \pm 1, \pm 2, \cdots$. Then*

$$u(z) = c_1 e^{\pi x} \sin \pi y + c_2 e^{-\pi x} \sin \pi y$$

for some real constants c_1 and c_2.

Of course, the above results hold for complex-valued $u(z)$.

2. **Proof of Theorem 1.** Let $f(z)$ be an entire function with $\operatorname{Re} f = u$ and let $F(z) = f(z) + [f(\bar{z})]^{-}$. Then $F(z)$ is an entire function of exponential type at most π. By the Paley-Wiener theorem, we have

$$F(z) = \int_{-\pi}^{\pi} e^{izt} \phi(t)\, dt.$$

Since $F(x) = 2u(x)$ is in $L^2(-\infty, \infty)$, $\phi(t)$ is in $L^2(-\pi, \pi)$ (cf. [**2**]). Hence,

$$\sum_{n=-\infty}^{\infty} u^2(n) = \frac{1}{4} \sum_{n=-\infty}^{\infty} F^2(n) = \frac{1}{4} \int_{-\pi}^{\pi} |\phi(t)|^2\, dt < \infty.$$

Similarly, we have $\sum u^2(n+i) < \infty$. Using Schwarz's inequality and the Plancherel theorem, we obtain

$$\begin{aligned} &\left\{ \sum_{n=-\infty}^{\infty} \left| \frac{u(n)}{2\pi} \int_{-\pi}^{\pi} \frac{\sinh t(1-y)}{\sinh t} e^{it(x-n)}\, dt \right| \right\}^2 \\ &\qquad \leqq \left\{ \frac{1}{4} \pi^2 \sum_{n=-\infty}^{\infty} u^2(n) \right\} \left\{ \sum_{n=-\infty}^{\infty} \left| \int_{-\pi}^{\pi} \left\{ \frac{\sinh t(1-y)}{\sinh t} e^{itx} \right\} e^{-itn}\, dt \right|^2 \right\} \\ &\qquad \leqq c_1 \int_{-\pi}^{\pi} \left| \frac{\sinh t(1-y)}{\sinh t} e^{itx} \right|^2 dt \\ &\qquad \leqq c_2 \int_{-\pi}^{\pi} \left| \frac{\sinh t(1-y)}{t} \right|^2 dt \leqq c_3 |1-y|^2 e^{2\pi|y|} \end{aligned}$$

for all $x+iy$. Similarly, we have

$$\sum_{n=-\infty}^{\infty} \left| \frac{u(n+i)}{2\pi} \int_{-\pi}^{\pi} \frac{\sinh ty}{\sinh t} e^{it(x-n)}\, dt \right| \leqq c_4 |y|\, e^{\pi|y|}.$$

Hence, the series in (1) converges uniformly in every strip $|y| \leqq K < \infty$ to a real-valued entire harmonic function $w(z)$ with

$$|w(x+iy)| \leqq (\alpha + \beta |y|) e^{\pi|y|} \tag{2}$$

for some constants α, β and for all $x+iy$.

Since $F(z)$ is an entire function of exponential type $\leqq\pi$ and its restriction to the real axis is in $L^2(-\infty, \infty)$, we have

$$\begin{aligned} u(x) = \frac{1}{2} F(x) &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{\sin \pi(x-n)}{x-n} F(n) \\ &= \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{\sin \pi(x-n)}{x-n} u(n) \\ &= \sum_{n=-\infty}^{\infty} \frac{u(n)}{2\pi} \int_{-\pi}^{\pi} e^{it(x-n)}\, dt = w(x) \end{aligned}$$

for all real x. Similarly, we can also conclude that $u(x+i)=w(x+i)$.

Now, let $h(z)$ be an entire function with $\operatorname{Re} h=u-w$. From (2) and Carathéodory's inequality, we have

$$h(z) = O(e^{(\pi+\varepsilon)|z|}) \tag{3}$$

for $0<\varepsilon<1$. Let $H(z)=h(z)+[h(\bar{z})]^{-}$. Then $H(x)=2 \operatorname{Re} h(x)=0$ for all real x so that $H(z)\equiv 0$ or

$$h(z) = -[h(\bar{z})]^{-}. \tag{4}$$

Similarly, let $G(z)=h(z+i)+[h(\bar{z}+i)]^{-}$. Then $G(x)=2 \operatorname{Re} h(x+i)=0$ for all real x so that $h(z+i)=-[h(\bar{z}+i)]^{-}$ for all z. Hence, by combining this with (4), we see that $h(z)$ has period $2i$. Now, it is well known that an entire function with period $2i$ satisfying (3) must be an exponential sum of the form $h(z)=\alpha+\beta e^{\pi z}+\gamma e^{-\pi z}$. That is, we have

$$\begin{aligned} &u(x+iy) - w(x+iy) \\ &\qquad = a + b_1 e^{\pi x} \cos \pi y + c_1 e^{\pi x} \sin \pi y + b_2 e^{-\pi x} \cos \pi y + c_2 e^{-\pi x} \sin \pi y \end{aligned}$$

for some real constants a, b_1, b_2, c_1, c_2. Since $u(x)=w(x)$ for all real x, $a=b_1=b_2=0$. From (2), we can conclude that

$$c_1 = \lim_{x\to\infty} e^{-\pi x} \frac{u(x+iy) - w(x+iy)}{\sin \pi y} = \lim_{x\to\infty} e^{-\pi x} \frac{u(x+iy)}{\sin \pi y}$$

for $0<y<1$. Similarly, for $0<y<1$,

$$c_2 = \lim_{x\to-\infty} e^{\pi x} \frac{u(x+iy)}{\sin \pi y}.$$

It is obvious that $c_1=c_2=0$ if $\tau<\pi$.

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