ADDITIONS AND CORRECTIONS TO "ON THE IDEAL STRUCTURE OF THE ALGEBRA OF RADIAL FUNCTIONS"

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ABSTRACT. The corrections and additions are made in the context of Hankel transforms which generalize the Fourier transforms of radial functions. The following question is studied: given two closed ideals $I_1$ and $I_2$ in the algebra of Hankel transforms such that both have the same spectrum and $I_1 \subseteq I_2$, when is there a closed ideal $I$ such that $I_1 \subseteq I \subseteq I_2$?

1. Introduction. Professor Aharon Atzmon has been kind enough to point out a counterexample to the theorem in the paper mentioned in the title. That theorem was to have been an analogue for radial functions to the following theorem of Helson [2] which is stated here for the algebra $L$ of integrable functions on $\mathbb{R}^n$ (see also [3, p. 183]).

THEOREM. Suppose $I_1 \subseteq I_2$ ($\subseteq$ denotes proper inclusion) where $I_1$ and $I_2$ are close ideals of $L$ such that

$$\{ y \mid \hat{f}(y) = 0 \text{ for every } f \in I_1 \} = \{ y \mid \hat{f}(y) = 0 \text{ for every } f \in I_2 \};$$

then there is another closed ideal $I$ of $L$ such that $I_1 \subseteq I \subseteq I_2$.

In this paper, we give the counterexample cited by Atzmon and a new theorem. The discussion will be carried out in the language of Hankel transforms; see [8, pp. 236–237] for definitions and notation.

When $n$ is a natural number and $\nu = (n - 2)/2$, $A_\nu$ is isometrically isomorphic to the algebra $L_\nu$ of radial integrable functions on $\mathbb{R}^n$ with the usual convolution. Indeed, it consists, essentially, of the restrictions of the members of $L_\nu$ to a ray starting at the origin; and in fact, the Hankel transforms of the functions in $A_\nu$ are exactly the restrictions to such a ray of the Fourier transforms of the functions in $L_\nu$.

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In the balance of the paper we assume $v$ is an arbitrary but fixed real number no smaller than $-\frac{1}{2}$. We let $k$ be the greatest integer not exceeding $v+\frac{1}{2}$. If $I$ is a closed ideal in $A_v$ we set $I = \{ f^t | f \in I \}$ and we say $f^t$ belongs to $I$ locally at $y$ if $y$ has a neighborhood in $[0, \infty)$ on which $f^t$ agrees with a member of $I$. (The term neighborhood will always refer to a bounded subset of $[0, \infty)$ open in the usual topology.)

The functions in $A_v$ all have $k$ continuous derivatives on $(0, \infty)$ (see [6] or [9]) and the functionals $f \mapsto f^{(i)}(y)$ are all continuous for $0 \leq j \leq k$ and $y > 0$. For $f \in A_v$ define

$$Z^{(i)}(f) = \{ y \mid f^{(i)}(y) = \cdots = f^{(j)}(y) = 0 \}$$

and if $I$ is an ideal, let

$$Z^{(i)}(I) = \bigcap_{f \in I} Z^{(i)}(f).$$

Since $f^{(i)}$ are continuous for $f \in A_v$, all of these sets are closed. We write $Z(I) = Z^{(0)}(I)$ and $Z(f) = Z^{(0)}(f)$, and call $Z(I)$ the spectrum of $I$. If $E$ is closed, the collection of all closed ideals with spectrum $E$ will be called $J(E)$. $I_0(E)$ will denote the smallest closed ideal containing all functions $f$ such that $E$ is interior to $Z(f)$. If $E$ is closed and $j = 0, 1, \cdots, k$, we define the closed ideals

$$I^{(j)}(E) = \{ f \mid f^{(i)}(y) = \cdots = f^{(j)}(y) = 0 \text{ all } y \in E \};$$

we write $I(E)$ for $I^{(0)}(E)$; $I(E)$ is the largest ideal in $J(E)$.

2. The counterexample. Let $E$ consist of a single positive real number $y$. Since the infinitely differentiable functions with compact support in $[0, \infty)$ are dense in $(A_v)^*$, it follows that the ideals

$$I(E) = I^{(0)}(E) \supseteq I^{(1)}(E) \supseteq \cdots \supseteq I^{(k)}(E)$$

are all distinct. It follows from [9, Lemma IV] that these are the only ideals of $A_v$ with spectrum $E$. Thus at least in the case $v \geq \frac{1}{2}$ when $E$ consists of a single positive real number the theorem of [7] is false.

In the balance of the paper, we will investigate the ideal structure of $A_v$ in an attempt to discover when the structure is discrete; i.e., when there are pairs of distinct closed ideals between which no closed ideals can be found.

3. Additional definitions and notation. If $E$ is a closed subset of $[0, \infty)$ then we say that $E$ has relative approximate identities if whenever $f \in A_v$ and $E \subseteq Z^{(k)}(f)$ there is a sequence of functions $\{ v_n \}$ such that $E$ is interior to $Z(v_n)$ and $\| v_n \ast f - f \| \to 0$ as $n \to \infty$. $\{ v_n \}$ is called an approximate identity for $f$ relative to $E$. 

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If \( E \) is any subset of \([0, \infty)\), \( \mathcal{C}E \) is its closure, and \( E' \), called the derived set of \( E \), is the set of limit points of \( E \).

We say a partially ordered set \( P \) has a discrete structure if given any \( x \in P \), there is a countable or finite subset \( B(x) \) such that

(a) if \( z \in P \) and \( z > x \) (\( z < x \)), then there is \( y \in B(x) \) such that \( z \geq y > x \) (\( z \leq y < x \));

(b) if \( y \in B(x) \) and \( y > x \) (\( y < x \)) there is no \( z \in P \) such that \( y > z > x \) (\( y < z < x \)).

Two partially ordered sets are isomorphic if there is an order-preserving bijection between them. If \( E \) is any subset of \([0, \infty)\), let \( \mathcal{W}(E) \) consist of all functions \( \phi \) such that \( \phi: E \to \{0, 1, \ldots, k\} \) and \( \phi(0) = 0 \) if \( 0 \in E \). We define \( \mathcal{I}(\phi) \) to be the collection of all \( f \in A_v \), such that

\[
\hat{f}(x) = \hat{f}^{(0)}(x) = \hat{f}(x) = \cdots = \hat{f}^{(\phi(x))}(x) = 0
\]

for every \( x \in E \). It is easy to see that \( \mathcal{I}(\phi) \subseteq \mathcal{I}(E) \). If \( \phi \) and \( \psi \) are in \( \mathcal{W}(E) \), we say that \( \phi \geq \psi \) if \( \phi(x) \leq \psi(x) \) for every \( x \in E \); if this is the case, it follows that \( \mathcal{I}(\phi) \supseteq \mathcal{I}(\psi) \); \( \mathcal{W}(E) \) is obviously discrete under this ordering whenever \( E \) is countable or finite.

**Remark.** The relative approximate identity is generally a weaker notion than that of approximate identity: a closed set \( E \) has an approximate identity if there is a sequence of functions \( \{v_n\} \) such that \( E \) is interior to \( Z(v_n) \) for \( n = 1, 2, \ldots \) and such that, if \( f \in \mathcal{I}(E) \), then \( \|f * v_n - f\| \to 0 \) as \( n \to \infty \) (cf. [3, pp. 48-51]). The existence of an approximate identity implies that \( \mathcal{I}(E) = I_0(E) \) so that \( \mathcal{S}(E) \) consists of exactly one ideal. The question of whether the converse of this is true is still open.

4. **Main result.** We are now in a position to state a theorem which gives some description of the ideal structure of \( A_v \).

**Theorem.** Let \( E \) be a closed subset of \([0, \infty)\), let \( E' \) be the derived set of \( E \), and let \( F \) be the set of isolated points of \( E \). Suppose \( E' \) has relative approximate identities. Then the mapping \( \phi \mapsto K(\phi) = I(\phi) \cap \mathcal{I}(E) \) is an isomorphism between \( \mathcal{W}(F) \) and \( \mathcal{S}(E) \), and so \( \mathcal{S}(E) \) is discrete.

The proof of the theorem will be given in §6. We note that there exists a sequence \( \{k_n\} \) of functions such that \( \|k_n * f - f\| \to 0 \) for every \( f \in A_v \), so the hypotheses above hold when \( E' \) is empty and we have the

**Corollary.** If \( E \) is a discrete subset of \([0, \infty)\) then \( \mathcal{S}(E) \) is isomorphic to \( \mathcal{W}(E) \), and so \( \mathcal{S}(E) \) is discrete.

5. **Lemmas.** The following lemmas describe the ideal structure of \( A_v \) and contain results which will be useful in proving the theorem.
Lemma 1. Suppose \( \hat{f}(0) = 0 \); then there is a sequence of functions \( \{v_n\} \) such that \( \hat{v}_n = 0 \) on a neighborhood of 0 and such that \( \|\hat{f} \ast v_n - \hat{f}\| \to 0 \).

This is Lemma 3.4.17 of [5], which is adapted from [3, pp. 48–51].

Lemma 2. If \( y_0 > 0 \), \( \mathcal{F}(y_0) \) consists of the \( k + 1 \) distinct ideals \( I^{(0)}(y_0), \ldots, I^{(k)}(y_0) \); in particular \( I_0(y_0) = I^{(k)}(y_0) \).

The lemma is proved easily with the use of Lemma IV of [9] and Lemma 4 of [7].

Lemma 3. (a) If \( f \in A_\nu \) and \( f \) does not vanish on some compact subset \( K \) of \( [0, \infty) \), then there is \( g \in A_\nu \) such that \( \hat{g}f = 1 \) on \( K \).

(b) If \( f \) and \( g \) are in \( A_\nu \) and \( Z(g) \) is interior to \( Z(f) \), then, if \( f \) has compact support, \( f = g \ast h \) for some \( h \in A_\nu \).

The lemma is easily proved using the techniques of [1, pp. 119–130].

Lemma 4. If \( I \) is a closed ideal of \( A_\nu \) and \( f \in A_\nu \) and if \( f \) is in \( \hat{I} \) locally at each point of \( [0, \infty) \), then \( f \in I \).

This can be proved by using the technique of [3, Lemma 6.2.6].

Lemma 5. If \( f \in A_\nu \) and \( I \) is a closed ideal of \( A_\nu \), then \( f \in \hat{I} \) locally at \( y_0 \) if either of the following two conditions hold:

(i) \( y_0 \notin Z(I) \).

(ii) \( y_0 \) is an interior point of \( Z(f) \).

Proof. (a) If \( y_0 \notin Z(I) \), then there is \( g \in I \) such that \( \hat{g}(y_0) \neq 0 \). The continuity of \( g \) and Lemma 3 ensure the existence of a function \( h \in A_\nu \) such that \( \hat{h}g = 1 \) on a neighborhood \( U \) of \( y_0 \). Then \( f \) coincides with \( f \hat{h}g \) in \( U \) and the latter function is the transform of \( (f \ast h) \ast g \) which is in \( I \), so \( f \) is in \( I \) locally at \( y_0 \).

(b) follows since \( I \) contains the constant zero function.

Lemma 6. If \( I \) is a closed ideal of \( A_\nu \), \( f \in A_\nu \) and \( Z(I) \subseteq Z^{(k)}(f) \), then the set

\[ C = \{y \mid \mathcal{F}f \text{ is not in } \hat{I} \text{ locally at } y\} \]

is closed, and if the derived set of \( Z(f) \) has relative approximate identities, then \( C \) has no isolated points; thus \( C \) is either empty or perfect.

Proof. The complement of \( C \) is open, so \( C \) is closed. Suppose by way of contradiction that \( y_0 \) is an isolated point of \( C \). Let \( W \) be a neighborhood of \( y_0 \) which contains no other point of \( C \) and let \( h \in A_\nu \), be such that \( \hat{h} = 1 \) on a neighborhood of \( y_0 \) and \( \hat{h} = 0 \) off of a compact subset contained in \( W \). Since \( y_0 \in C \), then \( y_0 \in Z(I) \) by Lemma 5, so \( y_0 \in Z^{(k)}(f) \). We treat two cases depending on whether or not \( y_0 \) is an isolated point of \( Z(f) \).
Suppose \( y_0 \) is a limit point of \( Z(f) \); a repeated application of Rolle's theorem shows that the derived set of \( Z(f) \) is contained in \( Z^{(k)}(f) \). Let \( \{v_n\} \) be an approximate identity for \( f \) relative to the derived set of \( Z(f) \); then \( v_n \) vanishes in a neighborhood of \( y_0 \). We claim \( f_n = v_n \ast h \ast f \) belongs to \( \mathcal{I} \) locally at every point of \([0, \infty)\). To see this, note \( y_0 \) is interior to \( Z(v_n) \) and if \( y \notin W \), \( y \) is interior to \( Z(h) \) so if \( y \notin W \) or \( y = y_0 \), \( y \) is interior to \( Z(f_n) \), and, by Lemma 5, \( f_n \) belongs to \( \mathcal{I} \) locally at \( y \). By assumption, \( f \) belongs to \( \mathcal{I} \) locally at every point of \( W - \{y_0\} \), hence so does \( f_n \). Thus \( f_n \) belongs to \( \mathcal{I} \) locally at every point of \([0, \infty)\), so \( f_n \in \mathcal{I} \) by Lemma 4. Since \( v_n \ast f \rightarrow f \), \( f_n \rightarrow h \cdot f \), and since \( \mathcal{I} \) is closed, \( h \cdot f \in \mathcal{I} \). But \( h \cdot f = f \) on a neighborhood of \( y_0 \), so \( f \) belongs to \( \mathcal{I} \) locally at \( y_0 \), whence \( y_0 \notin C \) which is a contradiction.

Suppose now that \( y_0 \) is an isolated point of \( Z(f) \). If \( y_0 = 0 \) we use Lemma 1 and proceed as above. If \( y_0 > 0 \), let \( W \) and \( h \) be chosen as above to satisfy the additional condition that \( \partial W \cap Z(f) = \{y_0\} \). It is an easy matter to construct a function \( g \) such that \( \hat{g} = f \) on \( W \) and such that \( Z(g) = \{y_0\} \). Then the closed ideal generated by \( g \) must be \( I^{(k)}(y_0) \) by Lemma 2. Now, also by Lemma 2, \( I^{(k)}(\{y_0\}) = I_0(\{y_0\}) \) so we can find \( g_n \) such that \( y_0 \) is interior to \( Z(g_n) \), \( \hat{g}_n \) has compact support, and \( \|g_n - g\| \rightarrow 0 \). It follows from Lemma 3 that there are functions \( v_n \) such that \( g_n = v_n \ast g \). In particular, each \( \hat{v}_n \) vanishes on a neighborhood of \( y_0 \), and on \( W \), \( \hat{v}_n \hat{f} = \hat{v}_n \hat{g} \). We can now show, as in the first part of the proof, that \( f_n = h \hat{v}_n \hat{f} \) belongs to \( \mathcal{I} \) locally at every point \( y \) of \([0, \infty)\), so \( f_n \in \mathcal{I} \) and finally \( f_n \hat{h} \hat{v}_n = \hat{g} \hat{h} \hat{v}_n \) and \( \|g \ast v_n - g\| \) so \( g \ast h \in \mathcal{I} \) but \( \hat{g} \hat{h} = f \) on a neighborhood of \( y_0 \); thus \( f \) is in \( \mathcal{I} \) locally at \( y_0 \) which is a contradiction.

6. Proof of theorem. We first show that the mapping is injective. Suppose \( \phi \) and \( \psi \) are in \( \mathcal{W}(\mathcal{F}) \). If \( \phi \neq \psi \) then \( \phi(x) \neq \psi(x) \) for some \( x \in F ; x \neq 0 \), since if \( 0 \in F \), \( \phi(0) = 0 = \psi(0) \). Let \( W \) be a neighborhood of \( x \) such that \( W \cap E = \{x\} \). Assume \( \phi(x) < \psi(x) \) and let \( g \) be a function with compact support and infinitely many derivatives such that

\[
g = 0 \quad \text{off } W,
\]

\[
g(x) = g'(x) = \cdots = g^{(\phi(x))}(x) = 0 \quad \text{and} \quad g^{(\psi(x))}(x) \neq 0.
\]

Then \( f = g \in A \), as can be seen by repeated integration by parts and, in fact \( f = g \) and \( f \in \mathcal{K}(\phi) \), but \( f \notin \mathcal{K}(\psi) \), so \( \mathcal{K}(\phi) \neq \mathcal{K}(\psi) \).

Now suppose \( f \in \mathcal{F}(E) \); we define \( \phi \) by setting \( \phi(0) = 0 \) if \( 0 \in F \), and \( \phi(y) = \max\{j | y \in Z^{(j)}(I)\} \) if \( y \in F \) and \( y \neq 0 \). Then \( I \subseteq \mathcal{K}(\phi) \). Suppose \( f \in \mathcal{K}(\phi) \), then \( E \subseteq Z^{(k)}(f) \) \( \{v_n\} \) be an approximate identity for \( f \) relative to \( E' \).

We claim \( f \hat{v}_n \) is contained in \( \mathcal{I} \) locally at every point of \([0, \infty)\). There are three cases: (i) \( y \notin E \), (ii) \( y \in F \), (iii) \( y \in E' \). Case (i) is disposed of by
Lemma 5. To handle case (ii) let $W$ be a neighborhood of $y$ such that $W \cap E = \{y\}$, let $m = \max\{j \mid f \in I^{(j)}(\{y\})\}$; then there is $g \in I$ such that $Z(g) \cap W = \{y\}$, $g(y) = \cdots = g^{(m)}(y) = 0$, and, unless $m = k$, $g^{(m+1)}(y) \neq 0$; then $(I^{(m)}(\{y\}))^{\wedge}$ coincides on $W$ with the closed ideal generated by $g$ because of Lemma 2. But $f \ast v_n \in I^{(m)}(\{y\})$ so $f \ast v_n$ coincides with a function from $I$ on $W$. Finally, if $y \in E'$, $y$ is interior to $Z(v_n)$ so $f \ast v_n$ belongs to $I$ locally at $y$. We have proved our claim so $f \ast v_n \in I$ by Lemma 4, but $\|f \ast v_n - f\| \to 0$ and $I$ is closed, so $f \in I$, therefore $I = K(\phi)$.

That the mapping is order preserving is obvious, so the proof is complete.

7. Additional remarks and open questions. We recall that when $n$ is a natural number and $v = (n - 2)$ the algebras $L_r$ and $A_r$ are isomorphic so our results give information about the ideal structure of $L_r$. Moreover to each ideal $L_r$ corresponds a unique rotation invariant ideal of $L$ (see [7, Lemma 4]), so we also get information about that class of ideals.

We conclude with some remarks and questions about the ideal structure of $A_r$ and relative approximate identities.

1. The hypothesis in the theorem that $E$ have approximate relative identities was chosen to assure that $I_0(E) = I^{(k)}(E)$. Is the converse true? In particular, if $E$ consists of a single point $y_0 > 0$, does $E$ have a relative approximate identity? (With appropriate change in the definition, the question can be asked when $y_0 = 0$. The answer to this is "yes" and can be found in [5, Theorem 3.3.2].) Might it be the case that there is a fixed sequence $\{v_n\}$ of functions in $A_r$ such that $y_0$ is interior to $Z(v_n)$ and $\|v_n \ast f - f\| \to 0$ as $n \to \infty$ for every $f \in I^{(k)}(\{y_0\})$?

2. If $I_0(E) \neq I^{(k)}(E)$ must there be an ideal between them? Can the ideal structure be dense in the sense that, if $I_1 \subset I_2$ are distinct ideals in $I(E)$, then there is an ideal $J$ such that $I_1 \subset J \subset I_2$, at least if $I_2 \subset I^{(k)}(E)$?

3. We observe that, if $E$ is closed, then $E' \subset Z^{(k)}(I)$ for $I \in \mathcal{I}(E)$. What can be concluded about the structure of $\mathcal{I}(E)$ given the structure of $\mathcal{I}(E')$? If $F$ is the set of isolated points of $E$ then does the structure of $\mathcal{I}(E)$ above $I^{(k)}(E)$ become complicated if that between $I_0(E)$ and $I^{(k)}(E)$ does?

To be exact, we ask if the following sort of theorem might hold: "If $I \in \mathcal{I}(E)$ there is a unique $J \in \mathcal{I}(E')$ and $\phi \in W(F)$ such that $I = J \cap I(\phi)$."?

4. If $I^{(k)}(E) = I(E)$ does it follow that $I^{(k)}(E) = I_0(E)$? The converse is, of course, false by Lemma 2.

References


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