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ADDITIONS AND CORRECTIONS TO "ON THE IDEAL STRUCTURE OF THE ALGEBRA OF RADIAL FUNCTIONS"

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ABSTRACT. The corrections and additions are made in the context of Hankel transforms which generalize the Fourier transforms of radial functions. The following question is studied: given two closed ideals I_1 and I_2 in the algebra of Hankel transforms such that both have the same spectrum and $I_1 \subset I_2$, when is there a closed ideal I such that $I_1 \subset I \subset I_2$?

1. **Introduction.** Professor Aharon Atzmon has been kind enough to point out a counterexample to the theorem in the paper mentioned in the title. That theorem was to have been an analogue for radial functions to the following theorem of Helson [**2**] which is stated here for the algebra L of integrable functions on R^n (see also [**3**, p. 183]).

THEOREM. *Suppose $I_1 \subset I_2$ ($\subset$ denotes proper inclusion) where I_1 and I_2 are close ideals of L such that*

$$\{y \mid \hat{f}(y) = 0 \textit{ for every } f \in I_1\} = \{y \mid \hat{f}(y) = 0 \textit{ for every } f \in I_2\};$$

then there is another closed ideal I of L such that $I_1 \subset I \subset I_2$.

In this paper, we give the counterexample cited by Atzmon and a new theorem. The discussion will be carried out in the language of Hankel transforms; see [**8**, pp. 236–237] for definitions and notation.

When n is a natural number and $\nu=(n-2)/2$, A_ν is isometrically isomorphic to the algebra L_r of radial integrable functions on R^n with the usual convolution. Indeed, it consists, essentially, of the restrictions of the members of L_r to a ray starting at the origin; and in fact, the Hankel transforms of the functions in A_ν are exactly the restrictions to such a ray of the Fourier transforms of the functions in L_r.

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In the balance of the paper we assume ν is an arbitrary but fixed real number no smaller than $-\frac{1}{2}$. We let k be the greatest integer not exceeding $\nu+\frac{1}{2}$. If I is a closed ideal in A_ν we set $\hat{I}=\{\hat{f}\mid f\in I\}$ and we say $\hat{f}$ belongs to $\hat{I}$ locally at y if y has a neighborhood in $[0, \infty)$ on which $\hat{f}$ agrees with a member of $\hat{I}$. (The term neighborhood will always refer to a bounded subset of $[0, \infty)$ open in the usual topology.)

The functions in A_ν all have k continuous derivatives on $(0, \infty)$ (see **[6]** or **[9]**) and the functionals $f\to\hat{f}^{(j)}(y)$ are all continuous for $0\leqq j\leqq k$ and $y>0$. For $f\in A_\nu$ define

$$Z^{(j)}(f) = \{y \mid \hat{f}(y) = \cdots = \hat{f}^{(j)}(y) = 0\}$$

and if I is an ideal, let

$$Z^{(j)}(I) = \bigcap_{f\in I} Z^{(j)}(f).$$

Since $\hat{f}^{(j)}$ are continuous for $f\in A_\nu$, all of these sets are closed. We write $Z(I)=Z^{(0)}(I)$ and $Z(f)=Z^{(0)}(f)$, and call $Z(I)$ the spectrum of I. If E is closed, the collection of all closed ideals with spectrum E will be called $\mathscr{I}(E)$. $I_0(E)$ will denote the smallest closed ideal containing all functions f such that E is interior to $Z(f)$. If E is closed and $j=0, 1, \cdots, k$, we define the closed ideals

$$I^{(j)}(E) = \{f \mid \hat{f}(y) = \cdots = \hat{f}^{(j)}(y) = 0 \text{ all } y\in E\};$$

we write $I(E)$ for $I^{(0)}(E)$; $I(E)$ is the largest ideal in $\mathscr{I}(E)$.

2. **The counterexample.** Let E consist of a single positive real number y. Since the infinitely differentiable functions with compact support in $[0, \infty)$ are dense in $(A_\nu)^\wedge$, it follows that the ideals

$$I(E) = I^{(0)}(E) \supset I^{(1)}(E) \supset \cdots \supset I^{(k)}(E)$$

are all distinct. It follows from **[9**, Lemma IV**]** that these are the only ideals of A_ν with spectrum E. Thus at least in the case $\nu\geqq\frac{1}{2}$ when E consists of a single positive real number the theorem of [7] is false.

In the balance of the paper, we will investigate the ideal structure of A_ν in an attempt to discover when the structure is discrete; i.e., when there are pairs of distinct closed ideals between which no closed ideals can be found.

3. **Additional definitions and notation.** If E is a closed subset of $[0, \infty)$ then we say that E has *relative approximate identities* if whenever $f\in A_\nu$ and $E\subseteq Z^{(k)}(f)$ there is a sequence of functions $\{v_n\}$ such that E is interior to $Z(v_n)$ and $\|v_n * f-f\|\to 0$ as $n\to\infty$. $\{v_n\}$ is called an *approximate identity for f relative to E.*

If E is any subset of $[0, \infty)$, $\mathscr{C}E$ is its closure, and E', called the derived set of E, is the set of limit points of E.

We say a partially ordered set P has a discrete structure if given any $x \in P$, there is a countable or finite subset $B(x)$ such that

(a) if $z \in P$ and $z > x$ $(z < x)$, then there is $y \in B(x)$ such that $z \geqq y > x$ $(z \leqq y < x)$;

(b) if $y \in B(x)$ and $y > x$ $(y < x)$ there is no $z \in P$ such that $y > z > x$ $(y < z < x)$.

Two partially ordered sets are isomorphic if there is an order-preserving bijection between them. If E is any subset of $[0, \infty)$, let $\mathscr{W}(E)$ consist of all functions ϕ such that $\phi: E \to \{0, 1, \cdots, k\}$ and $\phi(0) = 0$ if $0 \in E$. We define $I(\phi)$ to be the collection of all $f \in A_\nu$ such that

$$\hat{f}(x) = \hat{f}^{(0)}(x) = \hat{f}'(x) = \cdots = \hat{f}^{(\phi(x))}(x) = 0$$

for every $x \in E$. It is easy to see that $I(\phi) \in \mathscr{I}(\mathscr{C}E)$. If ϕ and ψ are in $\mathscr{W}(E)$, we say that $\phi \geqq \psi$ if $\phi(x) \leqq \psi(x)$ for every $x \in E$; if this is the case, it follows that $I(\phi) \supseteq I(\psi)$; $\mathscr{W}(E)$ is obviously discrete under this ordering whenever E is countable or finite.

Remark. The relative approximate identity is generally a weaker notion than that of approximate identity: a closed set E has an approximate identity if there is a sequence of functions $\{v_n\}$ such that E is interior to $Z(v_n)$ for $n = 1, 2, \cdots$ and such that, if $f \in I(E)$, then $\|f * v_n - f\| \to 0$ as $n \to \infty$ (cf. [**3**, pp. 48–51]). The existence of an approximate identity implies that $I(E) = I_0(E)$ so that $\mathscr{I}(E)$ consists of exactly one ideal. The question of whether the converse of this is true is still open.

4. Main result.

We are now in a position to state a theorem which gives some description of the ideal structure of A_ν.

Theorem. *Let E be a closed subset of* $[0, \infty)$, *let E' be the derived set of E, and let F be the set of isolated points of E. Suppose E' has relative approximate identities. Then the mapping $\phi \to K(\phi) = I(\phi) \cap I(E)$ is an isomorphism between $\mathscr{W}(F)$ and $\mathscr{I}(E)$, and so $\mathscr{I}(E)$ is discrete.*

The proof of the theorem will be given in §6. We note that there exists a sequence $\{k_n\}$ of functions such that $\|k_n * f - f\| \to 0$ for every $f \in A_\nu$, so the hypotheses above hold when E' is empty and we have the

Corollary. *If E is a discrete subset of* $[0, \infty)$ *then $\mathscr{I}(E)$ is isomorphic to $\mathscr{W}(E)$, and so $\mathscr{I}(E)$ is discrete.*

5. Lemmas.

The following lemmas describe the ideal structure of A_ν and contain results which will be useful in proving the theorem.

LEMMA 1. *Suppose* $\hat{f}(0)=0$; *then there is a sequence of functions* $\{v_n\}$ *such that* $\hat{v}_n=0$ *on a neighborhood of* 0 *and such that* $\|f * v_n - f\| \to 0$.

This is Lemma 3.4.17 of [**5**], which is adapted from [**3**, pp. 48–51].

LEMMA 2. *If* $y_0>0$, $\mathscr{I}(y_0)$ *consists of the* $k+1$ *distinct ideals* $I^{(0)}(y_0), \cdots, I^{(k)}(y_0)$; *in particular* $I_0(y_0)=I^{(k)}(y_0)$.

The lemma is proved easily with the use of Lemma IV of [**9**] and Lemma 4 of [**7**].

LEMMA 3. (a) *If* $f \in A_\nu$ *and* $\hat{f}$ *does not vanish on some compact subset* K *of* $[0, \infty)$, *then there is* $g \in A_\nu$ *such that* $\hat{g}\hat{f}=1$ *on* K.

(b) *If* f *and* g *are in* A_ν *and* $Z(g)$ *is interior to* $Z(f)$, *then, if* $\hat{f}$ *has compact support*, $f=g * h$ *for some* $h \in A_\nu$.

The lemma is easily proved using the techniques of [**1**, pp. 119–130].

LEMMA 4. *If* I *is a closed ideal of* A_ν *and* $f \in A_\nu$ *and if* $\hat{f}$ *is in* $\hat{I}$ *locally at each point of* $[0, \infty)$, *then* $f \in I$.

This can be proved by using the technique of [**3**, Lemma 6.2.6].

LEMMA 5. *If* $f \in A_\nu$ *and* I *is a closed ideal of* A_ν, *then* $\hat{f} \in \hat{I}$ *locally at* y_0 *if either of the following two conditions hold*:

(i) $y_0 \notin Z(I)$.

(ii) y_0 *is an interior point of* $Z(f)$.

PROOF. (a) If $y_0 \notin Z(I)$, then there is $g \in I$ such that $\hat{g}(y_0)\neq 0$. The continuity of g and Lemma 3 ensure the existence of a function $h \in A_\nu$ such that $\hat{h}\hat{g}=1$ on a neighborhood U of y_0. Then $\hat{f}$ coincides with $\hat{f}\hat{h}\hat{g}$ in U and the latter function is the transform of $(f * h) * g$ which is in I, so $\hat{f}$ is in $\hat{I}$ locally at y_0.

(b) follows since I contains the constant zero function.

LEMMA 6. *If* I *is a closed ideal of* A_ν, $f \in A_\nu$ *and* $Z(I) \subseteq Z^{(k)}(f)$, *then the set*

$$C = \{y \mid \hat{f} \text{ is not in } \hat{I} \text{ locally at } y\}$$

is closed, and if the derived set of $Z(f)$ *has relative approximate identities, then* C *has no isolated points*; *thus* C *is either empty or perfect.*

PROOF. The complement of C is open, so C is closed. Suppose by way of contradiction that y_0 is an isolated point of C. Let W be a neighborhood of y_0 which contains no other point of C and let $h \in A_\nu$ be such that $\hat{h}=1$ on a neighborhood of y_0 and $\hat{h}=0$ off of a compact subset contained in W. Since $y_0 \in C$, then $y_0 \in Z(I)$ by Lemma 5, so $y_0 \in Z^{(k)}(f)$. We treat two cases depending on whether or not y_0 is an isolated point of $Z(f)$.

Suppose y_0 is a limit point of $Z(f)$; a repeated application of Rolle's theorem shows that the derived set of $Z(f)$ is contained in $Z^{(k)}(f)$. Let $\{v_n\}$ be an approximate identity for f relative to the derived set of $Z(f)$; then v_n vanishes in a neighborhood of y_0. We claim $f_n=v_n * h * f$ belongs to $\hat{I}$ locally at every point of $[0, \infty)$. To see this, note y_0 is interior to $Z(v_n)$ and if $y \notin W$, y is interior to $Z(h)$ so if $y \notin W$ or $y=y_0$, y is interior to $Z(f_n)$, and, by Lemma 5, $\hat{f}_n$ belongs to $\hat{I}$ locally at y. By assumption, $\hat{f}$ belongs to $\hat{I}$ locally at every point of $W-\{y_0\}$, hence so does $\hat{f}_n$. Thus $\hat{f}_n$ belongs to $\hat{I}$ locally at every point of $[0, \infty)$, so $f_n \in I$ by Lemma 4. Since $v_n * f \to f$, $f_n \to h * f$, and since I is closed, $h * f \in I$. But $\hat{h}\hat{f}=\hat{f}$ on a neighborhood of y_0, so $\hat{f}$ belongs to $\hat{I}$ locally at y_0, whence $y_0 \notin C$ which is a contradiction.

Suppose now that y_0 is an isolated point of $Z(f)$. If $y_0=0$ we use Lemma 1 and proceed as above. If $y_0>0$, let W and h be chosen as above to satisfy the additional condition that $\mathscr{C}W \cap Z(f)=\{y_0\}$. It is an easy matter to construct a function g such that $\hat{g}=\hat{f}$ on W and such that $Z(g)=\{y_0\}$. Then the closed ideal generated by g must be $I^{(k)}(y_0)$ by Lemma 2. Now, also by Lemma 2, $I^{(k)}(\{y_0\})=I_0(\{y_0\})$ so we can find g_n such that y_0 is interior to $Z(g_n)$, $\hat{g}_n$ has compact support, and $\|g_n-g\| \to 0$. It follows from Lemma 3 that there are functions v_n such that $g_n=v_n * g$. In particular, each $\hat{v}_n$ vanishes on a neighborhood of y_0, and on W, $\hat{v}_n\hat{f}=\hat{v}_n\hat{g}$. We can now show, as in the first part of the proof, that $\hat{f}_n=\hat{h}\hat{v}_n\hat{f}$ belongs to $\hat{I}$ locally at every point y of $[0, \infty)$, so $f_n \in I$ and finally $\hat{f}\hat{h}\hat{v}_n=\hat{g}\hat{h}\hat{v}_n$ and $\|g * v_n \to g\|$ so $g * h \in I$ but $\hat{g}\hat{h}=\hat{f}$ on a neighborhood of y_0; thus $\hat{f}$ is in $\hat{I}$ locally at y_0 which is a contradiction.

6. **Proof of theorem.** We first show that the mapping is injective. Suppose ϕ and ψ are in $\mathscr{W}(F)$. If $\phi \neq \psi$ then $\phi(x) \neq \psi(x)$ for some $x \in F$; $x \neq 0$, since if $0 \in F$, $\phi(0)=0=\psi(0)$. Let W be a neighborhood of x such that $W \cap E=\{x\}$. Assume $\phi(x)<\psi(x)$ and let g be a function with compact support and infinitely many derivatives such that

$$g = 0 \qquad \text{off } W,$$
$$g(x) = g'(x) = \cdots = g^{(\phi(x))}(x) = 0 \quad \text{and} \quad g^{(\psi(x))}(x) \neq 0.$$

Then $f=\hat{g} \in A_\nu$ as can be seen by repeated integration by parts and, in fact $\hat{f}=g$ and $f \in K(\phi)$, but $f \notin K(\psi)$, so $K(\phi) \neq K(\psi)$.

Now suppose $I \in \mathscr{I}(E)$; we define ϕ by setting $\phi(0)=0$ if $0 \in F$, and $\phi(y)=\max\{j \mid y \in Z^{(j)}(I)\}$ if $y \in F$ and $y \neq 0$. Then $I \subseteq K(\phi)$. Suppose $f \in K(\phi)$, then $E' \subseteq Z^{(k)}(f)$; let $\{v_n\}$ be an approximate identity for f relative to E'.

We claim $\hat{f}\hat{v}_n$ is contained in $\hat{I}$ locally at every point of $[0, \infty)$. There are three cases: (i) $y \notin E$, (ii) $y \in F$, (iii) $y \in E'$. Case (i) is disposed of by

Lemma 5. To handle case (ii) let W be a neighborhood of y such that $W\cap E=\{y\}$, let $m=\max\{j|f\in I^{(j)}(\{y\})\}$; then there is $g\in I$ such that $Z(g)\cap W=\{y\}$, $\hat{g}(y)=\cdots=\hat{g}^{(m)}(y)=0$, and, unless $m=k$, $\hat{g}^{(m+1)}(y)\neq 0$; then $(I^{(m)}(\{y\}))^\wedge$ coincides on W with the closed ideal generated by g because of Lemma 2. But $f*v_n\in I^{(m)}(\{y\})$ so $\hat{f}\hat{v}_n$ coincides with a function from $\hat{I}$ on W. Finally, if $y\in E'$, y is interior to $Z(v_n)$ so $f*v_n$ belongs to I locally at y. We have proved our claim so $f*v_n\in I$ by Lemma 4, but $\|f*v_n-f\|\to 0$ and I is closed, so $f\in I$, therefore $I=K(\phi)$.

That the mapping is order preserving is obvious, so the proof is complete.

7. **Additional remarks and open questions.** We recall that when n is a natural number and $\nu=\frac{1}{2}(n-2)$ the algebras L_r and A_ν are isomorphic so our results give information about the ideal structure of L_r. Moreover to each ideal L_r corresponds a unique rotation invariant ideal of L (see [7, Lemma 4]), so we also get information about that class of ideals.

We conclude with some remarks and questions about the ideal structure of A_ν and relative approximate identities.

1. The hypothesis in the theorem that E have approximate relative identities was chosen to assure that $I_0(E)=I^{(k)}(E)$. Is the converse true? In particular, if E consists of a single point $y_0>0$, does E have a relative approximate identity? (With appropriate change in the definition, the question can be asked when $y_0=0$. The answer to this is "yes" and can be found in [5, Theorem 3.3.2].) Might it be the case that there is a fixed sequence $\{v_n\}$ of functions in A_ν such that y_0 is interior to $Z(v_n)$ and $\|v_n*f-f\|\to 0$ as $n\to\infty$ for every $f\in I^{(k)}(\{y_0\})$?

2. If $I_0(E)\neq I^{(k)}(E)$ must there be an ideal between them? Can the ideal structure be dense in the sense that, if $I_1\subset I_2$ are distinct ideals in $\mathscr{I}(E)$, then there is an ideal J such that $I_1\subset J\subset I_2$, at least if $I_2\subseteq I^{(k)}(E)$?

3. We observe that, if E is closed, then $E'\subseteq Z^{(k)}(I)$ for $I\in\mathscr{I}(E)$. What can be concluded about the structure of $\mathscr{I}(E)$ given the structure of $\mathscr{I}(E')$? If F is the set of isolated points of E then does the structure of $\mathscr{I}(E)$ above $I^{(k)}(E)$ become complicated if that between $I_0(E)$ and $I^{(k)}(E)$ does? To be exact, we ask if the following sort of theorem might hold: "If $I\in\mathscr{I}(E)$ there is a unique $J\in\mathscr{I}(E')$ and $\phi\in W(F)$ such that $I=J\cap I(\phi)$."

4. If $I^{(k)}(E)=I(E)$ does it follow that $I^{(k)}(E)=I_0(E)$? The converse is, of course, false by Lemma 2.

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