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AUTOMORPHISMS OF VON NEUMANN ALGEBRAS AS POINT TRANSFORMATIONS[1]

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ABSTRACT. Given a concrete separable C^*-algebra $\mathfrak{A}$ with unit and a faithful normal finite trace τ on $\mathfrak{A}''$, we introduce a notion of τ-almost every state on $\mathfrak{A}$ which has the proper relationship with Segal's noncommutative integration theory. We then prove that any *-automorphism of $\mathfrak{A}''$ is implemented by some point transformation in the state space of $\mathfrak{A}$, defined τ-almost everywhere. This generalizes the classical result of von Neumann-Maharam.

0. **Notation and introduction.** Throughout this paper, $\mathfrak{A}$ will denote a norm separable C^*-algebra[2] with unit I of operators on a Hilbert space $\mathscr{H}$. $\mathfrak{S}$ will denote the space of states on $\mathfrak{A}$, and τ a faithful normal finite trace on $\mathfrak{A}''$.

In §1 we define the notion of *τ-almost every state* in $\mathfrak{S}$ or *sets of full τ-measure* in $\mathfrak{S}$. In §2 it is shown that for each fixed element A in $\mathfrak{A}''$ we can define a set of full τ-measure, $S(A)$, in $\mathfrak{S}$ and a natural limiting procedure to define $\rho(A)$ for all ρ in $S(A)$. The assigned values $\{\rho(A) | \rho \text{ in } S(A)\}$ are unique to the extent that any two such procedures must agree on a common subset of full τ-measure. When used in the obvious way to define τ-almost uniform convergence for sequences in $\mathfrak{A}''$, it follows from Theorem B in the appendix[3] that our definition coincides with that of Segal [3, Definition 2.3].

In §3 we generalize to noncommutative[4] $\mathfrak{A}$ a result of Maharam [1, Lemma 4]; namely we show that any *-automorphism α of $\mathfrak{A}''$ (not necessarily leaving τ invariant) is induced by a transformation $\tilde{\alpha}$ of $\mathfrak{S}$

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[2] For definitions and basic results in operator theory see, for example, [2].

[3] Theorems A and B in the Appendix are slight modifications of proofs presented by E. Nelson in a course on functional analysis at Princeton University in the fall of 1971.

[4] We are motivated by the abelian case with: $\mathscr{H}=L_2[0,1]$; $\mathfrak{A}=C(X)$, the multiplication algebra of continuous functions on $[0, 1]$; $\mathfrak{A}''=L_\infty[0, 1]$, the multiplication algebra of (equivalence classes of) essentially bounded measurable functions, and $\tau(\cdot)=\int \cdot\, dx$.

defined τ-almost everywhere. $\tilde{\alpha}$ is unique to the extent that any two such transformations must agree on a common subset of full τ-measure.

1. **Definition of τ-almost every state.** A subset S of $\mathfrak{S}$ will be said to consist of *τ-almost every state* in $\mathfrak{S}$, or to be a *set of full τ-measure*, if there exists a sequence $\{P_n\}$ of projections P_n in $\mathfrak{A}''$ such that:

(a) $\{P_n\}$ is nondecreasing,

(b) $\tau(I - P_n) \to_n 0$,

(c) $S \supseteq S(\{P_n\}) \equiv \bigcup_n \overline{\text{convex hull } \{\omega_\psi \mid \psi \in \mathscr{H}, \|\psi\| = 1, P_n\psi = \psi\}}^{w^*}$,

where ω_ψ is the vector state on $\mathfrak{A}$ generated by ψ. A sequence $\{P_n\}$ of projections P_n from $\mathfrak{A}''$, satisfying (a) and (b), will be called an *exhaustion*.[5] The fundamental property of the class of sets of full τ-measure, which follows immediately from [**3**, Theorem 6], is that it is closed under countable intersection.

2. **Isolation of discontinuities.** At this point we introduce the main technical tool. It is Tomita's [**4**] generalization to the noncommutative case of the following fact: Given f in $\mathscr{L}_\infty[0, 1]$, we can isolate its discontinuities, in that given $\varepsilon > 0$ there exists a set $E(\varepsilon)$ with Borel measure less than ε off which f coincides with some g from $C[0, 1]$. Therefore for any pure state ρ on $C[0, 1]$, corresponding to a point mass δ_x with x in $[0, 1]\backslash E(\varepsilon)$, or in fact for any state ρ "concentrated" on $[0, 1]\backslash E(\varepsilon)$, we can unambiguously define $\rho(f)$ as $\rho(g)$. And if we are dealing with f and $\chi_{E(\varepsilon)}$ (the characteristic function for $E(\varepsilon)$) only as elements of $L_\infty[0, 1]$, the assignment of $\{\rho(f) | \rho$ "concentrated" on $[0, 1]\backslash E(\varepsilon)\}$ would be ambiguous only up to measure zero.

In the general (not necessarily commutative) case, let A in $\mathfrak{A}''$ be given. From [**4**, Theorem 6], for each $n = 1, 2, \cdots$, there exists A_n in $\mathfrak{A}$ and a projection P_n in $\mathfrak{A}''$ such that $\tau(I - P_n) < 1/2^n$ and $AP_n = A_nP_n$. Defining $R_n = \bigwedge_{i \geqq n} P_i$, it follows from [**2**, 2.1.5 and 2.5.2] that $\tau(I - R_n) \to_n 0$ so that $\{R_n\}$ is an exhaustion, and $AR_n = A_nR_n$. Therefore, for $\rho \in S(\{R_n\})$ we have $\rho = w^* \lim_\beta \sum_{\psi \in \beta} \omega_\psi$ with $R_{n_\rho}\psi = \psi$ for some n_ρ (where the coefficients for a convex sum of ω_ψ's are absorbed into the ψ's, and "β" labels the convex sums), and we define $\rho(A) = \rho(A_{n_\rho})$. If through different choices in the procedure we had arrived at a different exhaustion $\{Q_n\}$, and thus an assignment of values $\{\rho(A) | \rho$ in $S(\{Q_n\})\}$, since $\{R_n \wedge Q_n\}$ is an exhaustion the two assignments would agree on the common subset $S(\{R_n \wedge Q_n\})$ of full τ-measure.

[5] There is a more general definition for semifinite τ in the Appendix. The terminology is that of Nelson; see footnote 3.

In applications there may be a distinguished sequence $\{B_n\}$, with B_n in $\mathfrak{A}$, such that $B_n \to_n A$ in the strong operator topology. Then it is not difficult to prove using [**3**, Corollary 13.1] and Theorem B in the appendix that there exists a subset S of $S(\{R_n\})$, of full τ-measure, and a subsequence $\{B_{n_i}\}$ of $\{B_n\}$ such that for all ρ in S, $\lim_i \rho(B_{n_i})$ exists and equals $\rho(A)$ as we have already defined it.

3. **Application to $*$-automorphisms of $\mathfrak{A}''$.** We now use the procedure of §2 for our stated objective. Let α be a $*$-automorphism of $\mathfrak{A}''$ and $\{A_i\}$ a countable dense subset of $\mathfrak{A}$. Let $\{P_n(i)\}$ be an exhaustion appropriate for $\alpha^{-1}(A_i)$, as discussed in §2. From [**3**, Theorem 6] there exists a subset S of $\bigcap_i S(\{P_n(i)\})$ of full τ-measure, and definitions of $\rho[\alpha^{-1}(A_i)]$ for all ρ in S and all i. For ρ in S we define $[\tilde{\alpha}\rho](A_i)$ as $\rho[\alpha^{-1}(A_i)]$. It is easy to see that $\tilde{\alpha}(\rho)$ can be uniquely extended by continuity to a state $\tilde{\alpha}(\rho)$ in $\mathfrak{S}$. The map $\tilde{\alpha}: S \to \mathfrak{S}$ is the one claimed in §0. It is clear that any two maps constructed in the above manner must agree on a common subset of full τ-measure. We summarize this result in the following

THEOREM. *Let $\mathfrak{A}$ be a concrete, separable C^*-algebra with unit, and τ a faithful, normal, finite trace on $\mathfrak{A}''$. Then any $*$-automorphism of $\mathfrak{A}''$ is implemented by some point transformation of the state space of $\mathfrak{A}$, defined τ-almost everywhere. The point transformation is unique to the extent that any two such transformations must coincide τ-almost everywhere.*

Appendix. Throughout this appendix M will denote a countably decomposable semifinite von Neumann algebra on a Hilbert space $\mathscr{H}$. We define an *exhaustion* as a sequence $\{P_n\}$ of projections P_n from M such that

(a′) $(I-P_n)$ is eventually finite with respect to M, and nonincreasing.

(b′) $(I-P_n) \to_n 0$ in the strong operator topology.

Note that if M possesses a faithful normal finite trace then this definition coincides with that of §1.

We say that a sequence $\{A_n\}$, with A_n in M, "converges nearly everywhere to zero in the sense of Segal" if given $\varepsilon > 0$ there exists an exhaustion $\{P_n(\varepsilon)\}$ such that $\|A_n P_n(\varepsilon)\| < \varepsilon$ for all n. (This is a trivial modification of Segal's definition [**2**, Definition 2.3] which eliminates the significance of any finite number of terms of the sequence.)

THEOREM A (NELSON).[3] *A sequence $\{A_n\}$, with A_n in M, converges nearly everywhere to zero in the sense of Segal if and only if there exists an exhaustion $\{P_n\}$ such that $\|A_n P_j\| \to_n 0$ for all j.*

PROOF. "*If*" *direction*. Let $\varepsilon > 0$ be given. For each n there exists a subsequence $\{P_{n_i}\}$ such that $\|A_n P_{n_i}\| < \varepsilon$ for all i. The result then follows from [3, Theorem 6].

"*Only if*" *direction*. Let $\varepsilon = 1/n$, and let $\{P_j\}$ be an exhaustion for the strongly dense domain[6] given by the intersection over n of those defined by $\{P_j(1/n)\}$. Note from the proof of [3, Theorem 6] that for each j and k there exists $n(j;k)$ such that $P_j \leqq P_{n(j;k)}(1/k)$. Therefore, if $n \geqq n(j;k)$ we have

$$\|A_n P_j\| \leqq \|A_n P_{n(j;k)}(1/k)\| \leqq \|A_n P_n(1/k)\| < 1/k. \qquad \text{Q.E.D.}$$

THEOREM B (NELSON).[3] *A sequence $\{A_n\}$, with A_n in M, converges nearly everywhere to zero in the sense of Segal if and only if there exists an exhaustion $\{P_j\}$ such that $\|P_j A_n P_j\| \to_n 0$ for all j.*

PROOF. "*Only if*" *direction*. Obvious from Theorem A.

"*If*" *direction*. Let $\varepsilon > 0$ be given. Choose a faithful normal semifinite trace ϕ on M^+ such that $\phi(I - P_1) < \infty$. (This is easily seen to be possible, for example by using countable decomposability in the proof of [2, 2.5.3].) Then $\phi(I - P_j) < \infty$ for all j. By taking a subsequence if necessary we may assume that $\phi(I - P_j) < 1/2^j$. Define $\mathscr{H}_j = P_j \mathscr{H}$ and $L_j = \mathscr{H}_j \bigcap_{k \geqq j} A_k^{-1}(\mathscr{H}_k)$. It is not difficult to see that $\{L_j\}$ is a nondecreasing sequence of closed linear subspaces of $\mathscr{H}$, and that if we define Q_j as the projection onto L_j then Q_j is in M. Using [3, Lemma 3.1] and [2, 2.1.5] we see that

$$\phi(I - Q_j) \leqq 1/2^j + \sum_{k \geqq j} 2/2^j \xrightarrow[j]{} 0.$$

Therefore $\{Q_j\}$ is an exhaustion. Since $A_n Q_n = P_n A_n Q_n$ and $Q_n \leqq P_n$ for all n, we have $\|A_n Q_n\| = \|P_n A_n Q_n\| \leqq \|P_n A_n P_n\| < \varepsilon$ for all n. Q.E.D.

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[6] The notation is that of [3, Definition 2.1].