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A METRIC CHARACTERIZING ČECH DIMENSION ZERO

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ABSTRACT. In this paper we prove the following: a metrizable space (X, τ) has (Čech) dimension zero if and only if there is a metric for X, generating the topology τ, taking values in some subset of the nonnegative real numbers with 0 as its only cluster point.

It is possible to categorize some classes of metrizable spaces using the existence of metrics with special properties. See for example J. de Groot **[1]**, J. Nagata **[2]**, and L. Janos **[4]**. One may also obtain classifications depending on the existence of metrics with values in subsets of the real numbers. For instance, a metrizable space has a metric with values in the nonnegative integers if and only if it is regular A space, **[5]**. In this note we extend the classification to those spaces having metrics with values in a subset of the positive real numbers with 0 as the only cluster point.

Let R be the set of real numbers and N the positive integers.

DEFINITION. We say (X, τ) is F-metrizable if there exists a metric ρ for X, generating the topology τ, such that for all $(x, y) \in X \times X$, $\rho(x, y) \in F \subset [0, \infty)$. We call such a ρ an F-metric. For example, every metrizable space is F-metrizable with $F = [0, 1]$.

Before coming to the main theorem we prove a special case of it:

THEOREM 1. *Let (X, τ) be metrizable and let $F = \{0\} \cup \{1/3^n \mid n \in N\}$. The following are equivalent*:

(i) *X is F-metrizable*,

(ii) *X has Čech dimension zero*, *i.e.* Ind $X = 0$.

PROOF. "(ii)$\Rightarrow$(i)". Let $B(m)$ be the Baire space of weight m, that is, the cartesian product of a countable family of discrete spaces each of cardinality m.

We will show first that $B(m)$ is F-metrizable.

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If x and y are distinct points in $B(m)$ let $f(x, y)=\min\{k|x_k\neq y_k\}$ and define

$$\rho(x, y) = 3^{-f(x,y)} \quad \text{when } x \neq y,$$
$$= 0 \quad \text{otherwise.}$$

Then ρ is an F-metric for $B(m)$ generating the topology. Thus $B(m)$ is F-metrizable. Now, if X has Čech dimension zero, we can imbed it in $B(m)$ for some m **[6]**. The result now follows upon making the observation that subspaces of F-metrizable spaces are F-metrizable.

"(i)⇒(ii)". Let ρ be an F-metric for X generating the topology τ. We will show that ρ is a nonarchimedian metric. Let x, y and z be three distinct points in X and suppose that $\rho(x, y)=1/3^l$, $\rho(x, z)=1/3^m$ and $\rho(x, y)=1/3^n$. Then $1/3^l\leqq 1/3^m+1/3^n$. If $m\leqq l$ then $1/3^l\leqq 1/3^m\leqq \max\{1/3^m, 1/3^n\}$. On the other hand, if $m>l$ and $n>l$, suppose for instance that $m\geqq n>l$. Then

$$1/3^n + 1/3^n \geqq 1/3^m + 1/3^n \geqq 1/3^l.$$

Therefore $2/3^n\geqq 1/3^l$ and thus $2\geqq 3^{n-1}\geqq 3$, a contradiction. Thus either $m\leqq l$ or $n\leqq l$. In either case $1/3^l\leqq\max\{1/3^m, 1/3^n\}$. Therefore $\rho(x, y)\leqq \max\{\rho(x, z), \rho(z, y)\}$. This inequality holds also when the three points are not distinct. Thus (X, τ) has a nonarchimedian metric and hence (by **[1]**), Ind $X=0$.

The main result depends upon a special case of the following theorem of J. Nagata **[3]**, "A space X has (Čech) dimension $\leqq n$ if and only if we can introduce in X a topology preserving metric ρ such that all the open balls $B(x, \varepsilon)$ have boundaries of dimension $\leqq n-1$ and such that $\{B(x, \varepsilon)|x \in X\}$ is closure preserving for every $\varepsilon>0$."

Let $B=\{a_n\}\cup\{0\}$ where $\{a_n\}$ is an arbitrary strictly monotonically decreasing sequence of real numbers with limit zero and $H=\{1/n|n \in N\}\cup \{0\}$.

THEOREM 2. *Let (X, τ) be metrizable. Then the following are equivalent*:
(i) Ind $X=0$,
(ii) *X is B-metrizable*,
(iii) *X is H-metrizable.*

PROOF. "(ii)⇒(iii)" is immediate.

"(iii)⇒(i)". We will use Nagata's Theorem in the case $n=0$. Let ρ be a compatible metric with values in $\{0\}\cup\{1/n|n \in N\}=H$. Let $p \in X$ and $\varepsilon>0$ be given. We will show firstly that $B(p, \varepsilon)$ has an empty boundary. This follows from the following:

LEMMA. *$S(p, \varepsilon)=\{x|\rho(p, x)=\varepsilon\}$ is an open set.*

PROOF OF THE LEMMA. We need check this only when $S(p, \varepsilon)\neq\varnothing$. In this case necessarily $\varepsilon=1/m$ for some m in N. Let $x \in S(p, \varepsilon)$, $\delta=1/2m(m+1)$

and $y \in B(x, \delta)$. Suppose $y \notin S(p, \varepsilon)$. Then, either $\rho(p, y) < \varepsilon$ which implies $\rho(p, y) \leqq 1/(m+1)$, or $\rho(p, y) > \varepsilon$ and we must have $\rho(p, y) \geqq 1/(m-1)$ ($m \geqq 2$ necessarily). In the first case

$$\rho(p, x) \leqq \rho(p, y) + \rho(y, x) \leqq \frac{1}{m+1} + \delta$$
$$= \frac{1}{m+1} + \frac{1}{2m(m+1)} < \frac{1}{m} = \varepsilon,$$

a contradiction. If, on the other hand, $\rho(p, y) \geqq 1/m-1$, then

$$\frac{1}{m-1} \leqq \rho(p, y) \leqq \rho(p, x) + \rho(x, y) < \frac{1}{m} + \frac{1}{2m(m+1)}.$$

But this is true if and only if $1/(m-1) - 1/m < \frac{1}{2}(1/m - 1/(m+1))$, again a contradiction. Thus if $y \in B(x, \delta)$ then $y \in S(p, \varepsilon)$. Therefore $S(p, \varepsilon)$ is open. This completes the proof of the lemma.

Suppose now that $x \in \mathrm{Cl}(B(p, \varepsilon)) \backslash B(p, \varepsilon)$. Then $S(p, \varepsilon)$ is a neighbourhood of x not meeting $B(p, \varepsilon)$. Thus $B(p, \varepsilon)$ is closed and open and hence has an empty boundary.

We will now show that, for fixed $\varepsilon > 0$, the family $\{B(p, \varepsilon) \mid p \in X\}$ is closure preserving.

Let $A \subseteq X$ be an arbitrary subset. Then

$$\bigcup_{a \in A} \overline{B(a, \varepsilon)} \subset \overline{\bigcup_{a \in A} B(a, \varepsilon)}.$$

Suppose now that $x \in \mathrm{Cl}(\bigcup_{a \in A} B(a, \varepsilon)) \backslash \bigcup_{a \in A} B(a, \varepsilon)$. Then for all $a \in A$, $\rho(a, x) \geqq \varepsilon$. Thus $\rho(A, x) \geqq \varepsilon$.

Also, there is a sequence $\{b_n\} \subset X$ and a sequence $\{a_n\} \subset A$ such that $\rho(x, b_n) < 1/n$ and $\rho(b_n, a_n) < \varepsilon$. Thus $\rho(x, a_n) < \varepsilon + 1/n$ and hence $\rho(A, x) = \varepsilon$.

We will now consider the possible values of ε.

Case 1. Suppose for some m in N, $1/m < \varepsilon < 1/(m-1) \leqq 1$. Then $B(x, 1/m) \cap A = \varnothing$ and $B(x, 1/(m-1)) \cap A \neq \varnothing$. Thus for all $a \in A$, $\rho(x, a) > 1/(m-1)$ and $\rho(x, A) > \varepsilon$, a contradiction.

Case 2. Suppose now that $\varepsilon = 1/m$ for some m in N. Then $\rho(x, A) = 1/m$. Let $\delta = \frac{1}{2}(1/m + 1/(m+1))$.

If $y \in B(x, \delta) \cap B(a, \varepsilon)$ for some a in A then

$$\frac{1}{m} \leqq \rho(x, a) \leqq \rho(x, y) + \rho(y, a)$$
$$< \frac{1}{2}\left(\frac{1}{m} - \frac{1}{m+1}\right) + \frac{1}{m+1} < \frac{1}{m}, \quad \text{a contradiction.}$$

Thus $B(x, \delta)\cap\{\bigcup_{a\in A} B(a, \varepsilon)\}=\varnothing$ contradicting the assumption that $x\in \mathrm{Cl}(\bigcup_{a\in A} B(a, \varepsilon))$. Thus

$$\bigcup_{a\in A} B(a, \varepsilon) = \overline{\bigcup_{a\in A} B(a, \varepsilon)}$$

and the family is closure preserving.

From these remarks and Nagata's Theorem it follows that $\operatorname{Ind} X=0$ proving (i).

"(i)$\Rightarrow$(ii)". Let $\{a_n\}$ be a strictly monotonically decreasing sequence of real numbers with limit zero. As in Theorem 1 the proof will follow from the proof for $B(m)$. Let x and y be in $B(m)$. Define $f(x, y)$ as in Theorem 1 and set

$$\begin{aligned}\rho(x, y) &= a_{f(x,y)} \quad \text{when } x \neq y,\\ &= 0 \qquad\quad \text{otherwise.}\end{aligned}$$

Let x, y and z be distinct points in $B(m)$. Then either $f(x, y)\geqq f(x, z)$ or $f(x, y)<f(x, z)$ in which case $f(x, y)=f(y, z)$. Thus in either case $f(x, y)\geqq \min\{f(x, z), f(y, z)\}$. Therefore $\rho(x, y)\leqq\max\{\rho(x, z), \rho(y, z)\}$ and thus ρ satisfies the triangle law. The other metric axioms and the fact that ρ is compatible with the product topology on $B(m)$ are easily checked. This completes the proof of the theorem.

Concluding remarks. If $F\subseteq [0, \infty)$ has 0 as its only cluster point F is a countable set and we may label the points of $F\cap[0, 1]$, $\alpha_1>\alpha_2>\alpha_3>\cdots> 0$. The sequence $\{a_n\}$ is monotonically decreasing and has limit 0. Thus we have shown:

Corollary. *The space (X, τ) has (Čech) dimension zero if and only if there exists a metric for X, generating the topology τ, having values in R^+ with 0 as the only cluster point of those values.*

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