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ON HARDY'S INEQUALITY AND LAPLACE TRANSFORMS IN WEIGHTED REARRANGEMENT INVARIANT SPACES

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ABSTRACT. Hardy's well-known inequality relating the norm of a function and the norm of its integral mean in the Lebesgue spaces $L^p(\mu)$, $d\mu(t)=t^{\sigma-1}\,dt$, is extended to the class of rearrangement invariant spaces $X(\mu)$. These spaces include, for example, the $L^p(\mu)$, the Lorentz and the Orlicz spaces. As an application, necessary and sufficient conditions are obtained for an operator related to the Laplace transform to be bounded as a linear operator between rearrangement invariant spaces of μ-measurable functions.

For $\sigma>0$, we write $d\mu(t)=t^{\sigma-1}\,dt$ and denote by $L^p(\mu)$ the space of μ-measurable functions on $(0,\infty)$ for which the norm

$$\|f\|_{p,\mu}=\begin{cases}\left(\int_0^\infty |f(t)|^p\,d\mu(t)\right)^{1/p}, & 1\leqq p<\infty,\\ \operatorname*{ess\,sup}_{t>0}|f(t)|, & p=\infty,\end{cases}$$

is finite, and if X and Y are Banach spaces, $[X, Y]$ will denote the space of bounded linear operators from X into Y. We abbreviate $[X,X]=[X]$. Let the operators P and P' be defined by

$$(Pf)(s)=\frac{1}{s}\int_0^s f(t)\,dt \quad\text{and}\quad (P'f)(s)=\int_s^\infty f(t)\,\frac{dt}{t}$$

whenever the required integrals exist for all $s>0$. Then Hardy's celebrated theorem [**4**, pp. 245–246] may be stated in the form:

THEOREM *Suppose* $1\leqq p<\infty$. *Then* $P\in[L^p(\mu)]$ *if* $p>\sigma$ *and* $P'\in[L^p(\mu)]$ *if* $\sigma>0$.

In this paper we determine necessary and sufficient conditions which allow the $L^p(\mu)$ spaces which appear in Hardy's theorem to be replaced by a function space of a more general class. The class of spaces with which we deal possess the property of rearrangement invariance and include, for

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example, the $L^p(\mu)$ spaces, the Lorentz and the Orlicz spaces. We apply our results to the Laplace transform considered as a linear operator between rearrangement invariant spaces.

Following the approach of [2] we assume that $X(m)$, m denoting Lebesgue measure on $(0, \infty)$, is a Banach space of Lebesgue measurable functions on $(0, \infty)$ whose norm $\|\cdot\|_{X(m)}$ is rearrangement invariant in the sense that two functions which are equimeasurable with respect to m have the same norm. The rearrangement invariant space $X(\mu)$ then consists of those μ-measurable functions on $(0, \infty)$ for which $f^* \in X(m)$ and the norm in $X(\mu)$ is given by $\|f\|_{X(\mu)} = \|f^*\|_{X(m)}$. Here, as usual, f^* denotes the nonnegative, nonincreasing rearrangement of f which is equimeasurable with f in the sense that

$$\mu\{t:|f(t)| > y\} = m\{t:f^*(t) > y\} \qquad (y > 0).$$

For more details see [**1**], [**2**].

The upper index α and the lower index β corresponding to the rearrangement invariant space X was defined by Boyd [**2**] in terms of the function $h(s, X, X)$, where for rearrangement invariant spaces X and Y, $h(s, X, Y)$ denotes the norm in $[X(m), Y(m)]$ of the dilation operator $E_s:(E_sf)(t)=f(st)$. Note that if $X \subseteq Y$, then $h(1, X, Y)$ is finite. This follows from the closed graph theorem and [**1**, Definition 1.1(iv)].

We have the following:

THEOREM 1. *Let X be a rearrangement invariant space with upper index α and lower index β. Then*

(i) $P \in [X(\mu)]$ *if and only if* $\alpha\sigma < 1$.

(ii) $P' \in [X(\mu)]$ *if and only if* $\beta > 0$.

THEOREM 2. *Let X and Y be rearrangement invariant spaces and let $h(s)=h(s, X, Y)$. Then*

(i) *The condition $X \subseteq Y$ is necessary for $P \in [X(\mu), Y(\mu)]$ and the condition $\int_0^1 h(s^\sigma)\, ds < \infty$ is sufficient for $P \in [X(\mu), Y(\mu)]$.*

(ii) *The condition $X \subseteq Y$ is necessary for $P' \in [X(\mu), Y(\mu)]$ and the condition $\int_1^\infty h(s^\sigma)\, ds/s < \infty$ is sufficient for $P' \in [X(\mu), Y(\mu)]$.*

For the Lorentz space $L^{p,q}$ the indices are given by $\alpha=\beta=1/p$, so in particular, taking $p=q$ in Theorem 1 we recover Hardy's theorem. The indices for the various Lorentz spaces $\Lambda(\varphi, p)$, $M(\varphi, p)$ and the Orlicz spaces $L_{M\Phi}$, L_Φ have been computed by Boyd [**1**]. We leave to the reader the application of those results to our Theorem 1.

For the particular case $\sigma=1$, Theorems 1 and 2 were obtained by Boyd [**1**] and applied to a study of the Hilbert transform. Here, we give one application of our results to a transform which is related to the Laplace transform $\mathscr{L}$. Let the transform T be given by

$$(Tf)(s) = \frac{1}{s}(\mathscr{L}f)\left(\frac{1}{s}\right) = \int_0^\infty e^{-t/s} f(t)\,\frac{dt}{s} \qquad (s > 0).$$

It is well known that $T \in [L^p(\mu)]$ if and only if $p > \sigma$, indeed, the case $\sigma = 1$ is suggested in [**5**, p. 397, Ex. 16] as an application of Hardy's theorem. We prove the following:

THEOREM 3. *Let X and Y be rearrangement invariant spaces. Then $T \in [X(\mu), Y(\mu)]$ if and only if $P \in [X(\mu), Y(\mu)]$.*

COROLLARY 1. *If X has upper index α, then $T \in [X(\mu)]$ if and only if $\alpha\sigma < 1$.*

COROLLARY 2. *$X \subseteq Y$ is a necessary condition and $\int_0^1 h(s^\sigma, X, Y)\,ds < \infty$ is sufficient for $T \in [X(\mu), Y(\mu)]$.*

COROLLARY 3. *If $\sigma < 1$, then $T \in [X(\mu), Y(\mu)]$ if and only if $X \subseteq Y$.*

Corollaries 1 and 2 follow immediately from the theorems, and according to [**1**, p. 605, Lemma 3.2] $sh(s, X, Y) \leqq h(1, X, Y)$ for $0 < s < 1$, so if $\sigma < 1$ and $X \subseteq Y$ we have

$$\int_0^1 h(s^\sigma, X, Y)\,ds \leqq h(1, X, Y)\int_0^1 s^{-\sigma}\,ds < \infty$$

and Corollary 3 then follows from Corollary 2.

The theorems depend on the following lemma which is adapted from §3 of [**1**] and which deals with operators of the following form: Let $a(t)$ be nonnegative and measurable on $(0, \infty)$. Define

$$(Kf)(s) = \int_0^\infty a(t) f(st)\,dt$$

and

$$(\tilde{K}f)(s) = \int_0^\infty \frac{1}{\sigma}\, t^{1/\sigma} a(t^{1/\sigma}) f(st)\,\frac{dt}{t}$$

whenever the required integrals exist for all $s > 0$.

LEMMA. *Suppose X and Y are rearrangement invariant spaces and K, $\tilde{K}$ are as defined above.*

(a) *$K \in [X(\mu), Y(\mu)]$ if and only if $\tilde{K} \in [X(m), Y(m)]$, indeed, K and $\tilde{K}$ have the same norm in the respective spaces.*

(b) *If $c = \int_0^\infty a(s)\,h(s^\sigma, X, Y)\,ds < \infty$, then $K \in [X(\mu), Y(\mu)]$ with $\|K\| \leqq c$.*

(c) *If $K \in [X(\mu), Y(\mu)]$ and $A(s) = \int_0^s a(t)\,dt$ then $A(s)h(s^\sigma, X, Y) \leqq \|K\|$.*

(d) *If* $a(t)>0$ *on a set of positive measure and* $K\in[X(\mu), Y(\mu)]$ *then* $X\subseteq Y$.

PROOF. Let $f\in X(m)$ and define, for $t>0, g(t)=(\tau f)(t)=f(t^\sigma/\sigma)$. Then, for each $y>0$,

$$\begin{aligned} m\{t: g^*(t)>y\} &= \mu\{t: |g(t)|>y\} \\ &= \mu\{t: |f(t^\sigma/\sigma)|>y\} = m\{t: |f(t)|>y\} \end{aligned}$$

so that for any rearrangement invariant space Z we have

$$\|\tau f\|_{Z(\mu)} = \|g\|_{Z(\mu)} = \|g^*\|_{Z(m)} = \|f\|_{Z(m)}. \tag{1}$$

Now,

$$\begin{aligned} (Kg)((\sigma s)^{1/\sigma}) &= \int_0^\infty a(t)g((\sigma s)^{1/\sigma}t)\,dt = \int_0^\infty a(t)f(st^\sigma)\,dt \\ &= \int_0^\infty \frac{1}{\sigma}\,t^{1/\sigma}a(t^{1/\sigma})f(st)\,\frac{dt}{t} = (\tilde{K}f)(s) \end{aligned}$$

that is, $K(\tau f)=\tau(\tilde{K}f)$ and hence, from (1),

$$\|\tilde{K}f\|_{Y(m)} = \|\tau(\tilde{K}f)\|_{Y(\mu)} = \|K(\tau f)\|_{Y(\mu)}$$

from which (a) follows. Now according to [**1**, Theorem 3.1], $\tilde{K}\in[X(m), Y(m)]$ whenever

$$\int_0^\infty a(s)h(s^\sigma, X, Y)\,ds = \int_0^\infty \frac{1}{\sigma}\,s^{1/\sigma}a(s^{1/\sigma})h(s, X, Y)\,\frac{ds}{s} < \infty,$$

so (b) follows from (a). Finally, (c) and (d) follow easily from (a) and [**1**, Lemma 3.3].

Note that the operators P, P', and T are, for appropriate choices of $a(t)$, of the form K in the lemma. In particular, Theorem 2 follows immediately from the lemma, and we now prove Theorems 1 and 3.

PROOF OF THEOREM 1. According to (a) of the lemma, $P\in[X(\mu)]$ if and only if $\tilde{P}\in[X(m)]$ where

$$(\tilde{P}f)(s) = \int_0^1 \frac{1}{\sigma}\,t^{1/\sigma}f(st)\,\frac{dt}{t},$$

so (i) follows from (50) of [**2**]. Again by the lemma, $P'\in[X(\mu)]$ if and only if $P'=\sigma\tilde{P}'\in[X(m)]$. Now if X' is the associate space of X with upper index α', then $P'\in[X(m)]$ if and only if $P\in[X'(m)]$, and since $\beta=1-\alpha'$, (ii) follows from (i).

PROOF OF THEOREM 3. Since

$$(P\,|f|)(s) = \int_0^1 |f(st)|\,dt \leqq e\int_0^\infty e^{-t}\,|f(st)|\,dt = e(T\,|f|)(s)$$

we get $P\in[X(\mu), Y(\mu)]$ whenever $T\in[X(\mu), Y(\mu)]$. On the other hand, $T|f|\leqq P|f|+Q|f|$ where

$$(Qf)(s)=\int_1^\infty e^{-t}f(st)\,dt \qquad (s>0),$$

so we need only show that $Q\in[X(\mu), Y(\mu)]$ whenever $P\in[X(\mu), Y(\mu)]$. Now if $P\in[X(\mu), Y(\mu)]$ then $X\subseteq Y$ by Theorem 2 so that

$$\int_1^\infty e^{-s}h(s^\sigma, X, Y)\,ds \leqq h(1, X, Y)\int_1^\infty e^{-s}\,ds<\infty$$

and $Q\in[X(\mu), Y(\mu)]$ by (b) of the lemma.

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