PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 39, Number 2, July 1973

DISTORTION THEOREMS FOR A SPECIAL CLASS OF ANALYTIC FUNCTIONS

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ABSTRACT. Sharp bounds are derived for the derivative of analytic functions of class P_α defined by the condition $|p(z)-1/2\alpha|\leqq 1/2\alpha$, $0\leqq\alpha\leqq 1$, $p(0)=1$. These results are improved for the class of functions with missing terms. Application is made to the class of functions with derivative $\in P_\alpha$ and the radius of convexity is determined for this class.

1. **Introduction.** In this paper sharp estimates are derived for $|p(z)|$, $|p'(z)|$ and $|p'(z)/p(z)|$ for functions $p(z)$, analytic in $|z|<1$, $p(0)=1$ assuming values in a disc whose boundary contains the origin. In §3 the results are sharpened for functions with missing initial terms. The methods used are largely based on the principle of subordination. The bounds obtained are related to corresponding estimates for classes considered by the author in a previous publication [6], functions bounded by the constant one in $|z|<1$ and functions whose real parts are bounded from below. In §4 the functions considered are a subclass of the class S of functions normalized by $f(0)=0, f'(1)=1$ with the value of $f'(z)$ restricted to a disc. The radius of convexity for this class, and for functions with missing terms is derived. The functions considered include as a special case the class of functions with positive real part; several classical results for this class follow from the theorems in this paper. The case of restriction to a disc of radius one also yields some known results [5].

2. **Distortion theorems for the class P_α.** Let P_α be defined as the class of analytic functions with expansion of the form $p(z)=1+a_1z+a_2z^2+\cdots$ in $|z|<1$ which satisfy in $|z|<1$ the inequality

$$|p(z)-1/2\alpha|\leqq 1/2\alpha, \quad \text{where } 0\leqq\alpha<1. \tag{1}$$

The cases $\alpha=0$ corresponding to $\operatorname{Re} p(z)>0$ and $\alpha=1/2$ implying values in a disc of radius 1 have been investigated by MacGregor ([4], [5]). The function $h(z)=1/p(z)$ will satisfy $\operatorname{Re} h(z)>\alpha$; some of the bounds were derived by the author [6] for this class.

Received by the editors May 11, 1972 and, in revised form, August 21, 1972 and October 11, 1972.

AMS (MOS) subject classifications (1970). Primary 30A32.

The univalent function $p^0(z)=(1-z)/(1+cz)$ where $c=1-2\alpha$, $-1<\varepsilon\leqq 1$, maps the interior of the unit disc onto the interior of the circle $|w-1/2\alpha|=1/2\alpha$ with $p^0(0)=1$. The function $p(z)$ satisfying (1) is subordinate to $p^0(z)$ and can therefore be written in the form $p(z)=[1-g(z)]/[1+cg(z)]$ where $|g(z)|\leqq|z|$. As a consequence we obtain

THEOREM 1. *For $p(z)\in P_\alpha$,*

$$(1-|z|)/(1+c|z|)\leqq \operatorname{Re} p(z)\leqq|p(z)|\leqq(1+|z|)/(1-c|z|),$$
$$c=1-2\alpha \text{ for } |z|<1.$$

Next a bound is derived for $|p'(z)|$ in terms of $p(z)$. The proof will be found in §3 for a more general case.

THEOREM 2. *Let $p(z)\in P_\alpha$; then*

$$|p'(z)|\leqq|1+cp(z)|^2/(1+c) \quad \textit{for } |z|\leqq[2]^{1/2}-1$$

and

$$|p'(z)|\leqq|1+cp(z)|^2(1+|z|^2)^2/4(1+c)|z|(1-|z|^2) \quad \textit{for } |z|>[2]^{1/2}-1.$$

THEOREM 3. *Let $p(z)\in P_\alpha$; then*

$$|p'(z)/p(z)|\leqq(1+c)/(1-|z|)(1+c|z|) \quad \textit{for } |z|<1. \tag{2}$$

PROOF. Let $h(z)=1/p(z)$; then $h(z)$ is analytic for $|z|<1$, $h(0)=1$ and it follows from inequality (1) that $h(z)$ satisfies $\operatorname{Re} h(z)>\alpha$, and $h'(z)/h(z)=-p'(z)/p(z)$. The bound (2) was proved by the author in Theorem 5 **[6]** for $h(z)$.

Theorem 3 will also be obtained as a special case of Theorem 6 but is included here on account of the simplicity of the proof. A result on the bound of $\operatorname{Re}\{zp'(z)/p(z)\}$ for $c\geqq 0$ equivalent to (2) was also obtained by Janowski **[3]** by means of a variational method.

THEOREM 4. *Let $p(z)\in P_\alpha$ with $0\leqq\alpha\leqq 1/2$; then*

$$|p'(z)|\leqq(1+c)/(1-c|z|)^2 \quad \textit{for } |z|\leqq\rho \tag{3a}$$

and

$$|p'(z)|\leqq\frac{(1+c)(1+|z|^2)^2}{4|z|(1-c)(1-|z|^2)(1+c|z|^2)} \quad \textit{for } |z|>\rho, \tag{3b}$$

where ρ is the largest positive root of the equation

$$cx^3+x^2(1-2c)+x(2-c)-1=0. \tag{4}$$

PROOF. From our representation for $p(z)$ we obtain

$$p'(z)=-g'(z)(1+c)/[1+cg(z)]^2.$$

The condition $\alpha\leqq 1/2$ implies $c\geqq 0$.

We have [**1**, pp. 18–19] $g'(z)=(1-|g(z)|^2)\phi(z)/(1-|z|^2)$ where $|\phi(z)|\leqq 1$. Writing $|\phi(0)|=a$ we obtain

$$|p'(z)| \leqq \frac{(1+c)(1-a|z|)(a+|z|)}{(1-ac|z|)^2(1-|z|^2)}. \tag{5}$$

By differentiating (5) with respect to a we find that the right hand side assumes its maximum value for

$$a = 1 + 2|z|^2 c - |z|^2/|z|(2 - c(1-|z|^2)). \tag{6}$$

The condition $a\leqq 1$ leads to the inequality $c|z|^3+|z|^2(1-2c)+|z|(2-c)-1\geqq 0$. Substitution of $a=1$ or the right hand side of (6) yields the estimates (3a) and (3b) respectively.

COROLLARY 1. *Let* $\operatorname{Re} p(z)>0$, *then* $|p'(z)|\leqq 2/(1-|z|)^2$ *for* $|z|<1$.

The solution of (4) for $c=1$ is $|z|=1$ and the bound (3a) holds in the entire disc. This bound was obtained by the author in [**6**] for the function $h(z)=1/p(z)$; the two classes coincide in this case.

COROLLARY 2. *Let* $|p(z)-1|\leqq 1$, *i.e.* $c=0$, *then the solution of* (4) *is* $\rho=\sqrt{2}-1$ *and the estimates* (3a) *and* (3b) *are identical with the known* [**1**] *bounds for* $|g'(z)|$ *since the transformation from* $p(z)$ *to* $g(z)$ *reduces to a translation of the plane.*

3. **Functions with missing coefficients.** In this section the subclass $P_{\alpha,n}$ of P_α considered will have an expansion

$$p(z) = 1 + a_n z^n + a_{n+1}z^{n+1} + \cdots, \qquad n \geqq 1.$$

Improved versions of Theorems 1 to 3 are obtained for the class $P_{\alpha,n}$.

Let $g(z)=[1-p(z)]/[1+cp(z)]=b_n z^n+\cdots$. It follows by the maximum modulus theorem that

$$|g(z)| \leqq |z|^n. \tag{7}$$

By subordination we obtain the bounds in

THEOREM 5. $(1-|z|^n)/(1+c|z|^n)\leqq \operatorname{Re} p(z)\leqq |p(z)|\leqq (1+|z|^n)/[1-c|z|^n]$.

The estimate for $|p'(z)|$ is given in the next theorem.

THEOREM 6. *Let* $p(z)\in P_{\alpha,n}$; *then*

$$|p'(z)| \leqq |1+cp(z)|^2\, n\,|z|^{n-1}/(1+c) \quad \textit{for } |z| \leqq ([1+n^2]^{1/2}-1)/n,$$

$$|p'(z)| \leqq \frac{|z|^{n-2}[4|z|^2 + n^2(1-|z|^2)^2]\,|1+cp(z)|^2}{4(1-|z|^2)(1+c)}$$

$$\textit{or } |z| > ([1+n^2]^{1/2}-1)/n.$$

Proof. The following bounds for functions $g(z)$ satisfying (7) were established in [**6**].

(8a) $$|g'(z)| \leqq n\,|z|^{n-1} \quad \text{for } |z| \leqq ([1+n^2]^{1/2}-1)/n,$$

(8b) $$|g'(z)| \leqq |z|^{n-2}\,[4\,|z|^2 + n^2(1-|z|^2)^2]/4(1-|z|^2) \qquad \text{for } |z| > [(1+n^2)^{1/2}-1]/n.$$

The result of the theorem follows by substitution of (8a) and (8b) in $p'(z) = -g'(z)(1+cp(z))^2/(1+c)$.

Theorem 2 corresponds to $n=1$.

Theorem 7. *Let $p(z) \in P_{\alpha,n}$; then*

(9) $$\left|\frac{p'(z)}{p(z)}\right| \leqq \frac{(1+c)n\,|z|^{n-1}}{(1+c\,|z|^n)(1-|z|^n)} \quad \textit{for } |z| < 1.$$

Proof. We have

$$p'(z)/p(z) = -g'(z)(1+c)/[1+cg(z)][1-g(z)]$$

where $|g(z)| \leqq |z|^n$.

Using again the notation and methods introduced in the proof of Theorem 4, we use the inequality

$$(1-|z|^2)\,|g'(z)|/|z|^{n-1} \leqq -a^2\,|z| + an(1-|z|^2) + |z|,$$

derived by the author in [**6**].

We can write $g(z) = z^n\phi(0)$, where $|\phi(0)| = a \leqq 1$.

To find the upper bound of $|p'(z)/p(z)|$ we need to find the smallest value of $D = |(1+cg(z))(1-g(z))|$. For $c \geqq 0$, $D = |1-g(z)(1-c)-cg^2(z)|$, both coefficients of the terms involving $g(z)$ are $\leqq 0$ and the minimum of D is obtained for $|g(z)| = a|z|^n > 0$. For $c < 0$ both factors in D are minimized for $g(z) > 0$ and we obtain for all c

(10) $$\frac{1-|z|^2}{|z|^{n-1}(1+c)}\left|\frac{p'(z)}{p(z)}\right| \leqq \frac{-a^2\,|z| + an(1-|z|^2) + |z|}{1 - a\,|z|^n\,(1-c) - a^2c\,|z|^{2n}} = \frac{-a^2\,|z| + Aa + |z|}{1 - aB - a^2C} = F(a),$$

where $A = n(1-|z|^2)$, $B = |z|^n(1-c)$, $C = c|z|^{2n}$, $A > 0$, $B \geqq 0$, $-1 < C < 1$. We wish to show that $F(a)$ increases monotonically for $0 < a < 1$.

The critical points of $F(a)$ are the roots of

$$x^2(|z|\,B + AC) - 2x\,|z|\,(1-C) + A + B\,|z| = 0.$$

The coefficient of x^2 is $|z|^{n+1}(1-c)+n(1-|z|^2)c|z|^{2n}$ which can be shown to be positive for $|z|<1$, and $-1<c\leqq 1$, leading to two positive roots whose product $(A+B|z|)/(AC+B|z|)>1$. The condition for the smaller root to be greater than 1, reduces to the condition

$$(11)\quad 2\,|z|^{n+1}\,(1-c)+n(1-|z|^2)(1+c\,|z|^{2n})-2\,|z|\,(1-c\,|z|^{2n})>0.$$

For the case $c=-1$, we obtain

$$(1-|z|^n)[-2\,|z|\,(1-|z|^n)+n(1-|z|^2)(1+|z|^n)]>0.$$

Let $\phi(|z|,n)=[-2|z|(1-|z|^n)+n(1-|z|^2)]$, $\phi(|z|,1)>0$, $\phi(|z|,n+1)>\phi(|z|,n)$. It follows that $\phi(|z|,n)+n|z|^n(1-|z|^2)>0$ and (11) is satisfied. A similar proof can be carried out for $c=1$, and since (11) is linear in c, it will hold for $-1<c<1$, and $F(a)<F(1)$ for $0<a<1$. The bound (9) follows by letting $a=1$ in (10).

This bound is sharp. It is assumed by the function $p(z)=(1-z^n)/(1+cz^n)$. Theorem 7 reduces to Theorem 3 for $n=1$.

Since $|h'(z)/h(z)|=|p'(z)/p(z)|$ for $h(z)=1/p(z)$ the bound (9) also holds for the class of functions $h(z)=1+a_nz^n+\cdots$ with $\mathrm{Re}(h(z))>\alpha$, and represents an extension of the author's Theorem 6 in **[6]** where (9) was shown to hold in a smaller disc.

4. **Distortion theorems for functions whose derivative belongs to P_α.** Let F_α denote the subclass of the class S of functions, analytic in $|z|<1$, $f(0)=0$ and $f'(0)=1$, whose derivative $f'(z)\in P_\alpha$. Since F_α is a subclass of functions with $\mathrm{Re}\,f'(z)>0$, $f(z)$ is schlicht if $f(z)\in F_\alpha$.

Estimates for $|f'(z), |f''(z)|$ and $|f''(z)/f'(z)|$ are obtained from Theorems 1, 2, 3 and 4 by identifying $f'(z)=p(z)$.

THEOREM 8. *Let $f(z)\in F_\alpha$, then*

$$(12)\quad \begin{aligned}&-|z|/c+((c+1)/c^2)\log(1+c\,|z|)\\&\qquad\leqq|f(z)|\leqq-|z|/c-((c+1)/c^2)\log(1-c\,|z|).\end{aligned}$$

These bounds are obtained by integration of the estimates of Theorem 1 applied to $p(z)=f'(z)$. Evaluating the limit $c\to 0$, the bounds (12) reduce to

$$|z|-|z|^2/2\leqq|f(z)|\leqq|z|+|z|^2/2,$$

an estimate established by MacGregor **[5]** for functions satisfying the inequality $|f'(z)-1|\leqq 1$. The estimate for the special case $c=1$, which means $\mathrm{Re}\,f'(z)>0$, was also proved by MacGregor in an earlier paper **[4]**.

COROLLARY. *Each function $f(z)\in F_\alpha$ assumes its values in the annulus*

$$-1/c+((c+1)/c^2)\log(1+c)<|z|<-1/c-((c+1)/c^2)\log(1-c).$$

THEOREM 9. *The radius of convexity of the class F_α is given by $r_0 = [(1+c)^{1/2}-1]/c$.*

PROOF. By application of Theorem 3 we obtain

$$\left|\frac{f''(z)}{f'(z)}\right| \leqq \frac{1+c}{(1-|z|)(1+c|z|)};$$

$f(z)$ maps $|z|<r$ onto a convex domain if $\mathrm{Re}\{z(f''(z)/f'(z))+1\}\geqq 0$, i.e. $|z|(1+c)/(1-|z|)(1+c|z|)\leqq 1$, i.e. $|z|<((1+c)^{1/2}-1)/c$.

For the function $f(z)=-z/c-((c+1)/c^2)\log(1-cz)$,

$$\mathrm{Re}\{z(f''(z)/f'(z))+1\}=0 \quad \text{for } z=r_0.$$

The known conditions of convexity ([4], [5]), $|z|<1/2$ and $|z|<\sqrt{2}-1$, are obtained for $c=0$ and $c=1$ respectively.

5. **Functions with derivative in class $P_{\alpha,n}$.** Sharper results can be obtained for a function of class S with expansion $f(z)=z+a_{n+1}z^{n+1}+a_{n+2}z^{n+2}\cdots$, $n\geqq 1$, and $f'(z)\in P_{\alpha,n}$. In this case Theorems 5, 6 and 7 can be applied to obtain estimates for $|f'(z)|$, $|f''(z)|$ and $|f''(z)/f'(z)|$. The bound for $|f''(z)/f'(z)|$ given by (9) will also be valid for the subclass of S whose derivative satisfies $\mathrm{Re} f'(z)>\alpha$.

THEOREM 10. *The radius of convexity r_0 for functions of class S with $f'(z)\in P_{\alpha,n}$ is given by*

$$r_0=\left\{\frac{([(n+1)+(n-1)c]^2+4c)^{1/2}-[(n+1)+(n-1)c]}{2c}\right\}^{1/n}. \tag{13}$$

PROOF. It follows from (9) that $|zf''(z)/f'(z)|\leqq 1$ if

$$(1+c)n|z|^n \leqq 1-(1-c)|z|^n-c|z|^{2n}. \tag{14}$$

The bound (13) corresponds to the positive solution for $|z|^n$ of condition (14).

Using the extremal function of Theorem 7 for $f'(z)$ it is seen that such functions have a point of inflection for $|z|=r_0$ and are not convex for any larger value of $|z|$.

For $c=0$ the known [5] solution $|z|=(n+1)^{-1/n}$ is obtained.

COROLLARY. *Let $f(z)\in S$ have the expansion $z+a_{n+1}z^{n+1}+\cdots$ and $\mathrm{Re}\{f'(z)\}>0$, then $f(z)$ will map the disc $|z|\leqq\{[n^2+1]^{1/2}-n\}^{1/n}$ onto a convex domain.*

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